Computational Geometry

Chapter 3

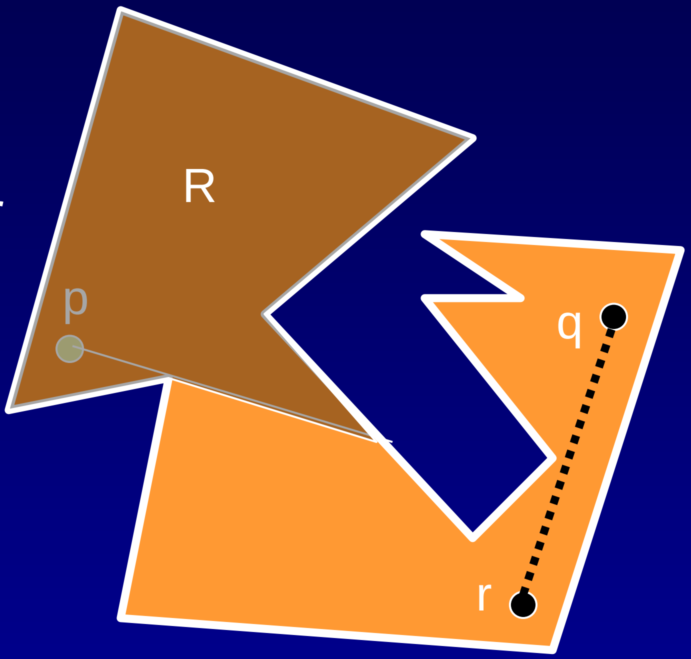
Polygons and Triangulation

On the Agenda

- The Art Gallery Problem
- Polygon Triangulation

Art Gallery Problem

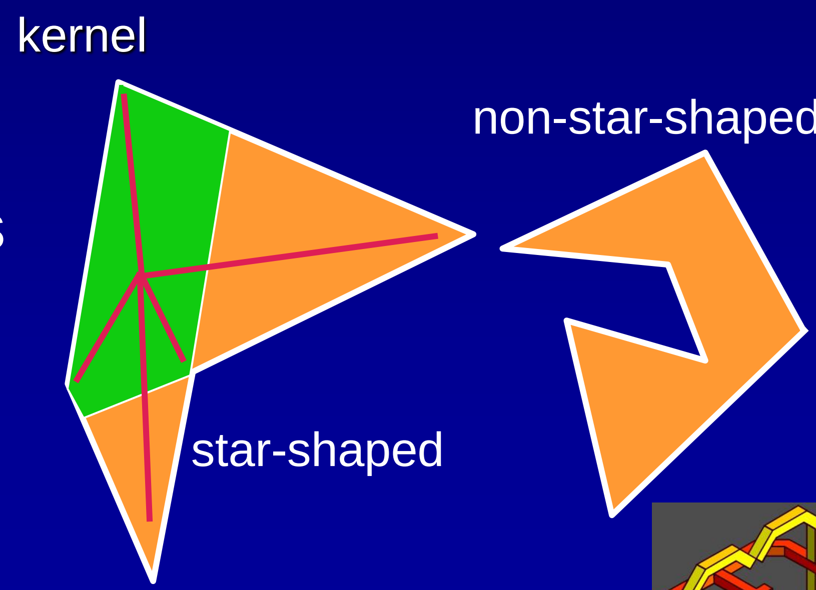
- Given a simple polygon *P*, say that two points *q* and *r* can *see* each other if the open segment *qr* lies entirely within *P*.
- A point *p guards* a region $R \subseteq P$ if *p* sees all points $q \in R$.
- Given a polygon *P*, what is the minimum number of guards required to guard *P*, and what are their locations?



Observations

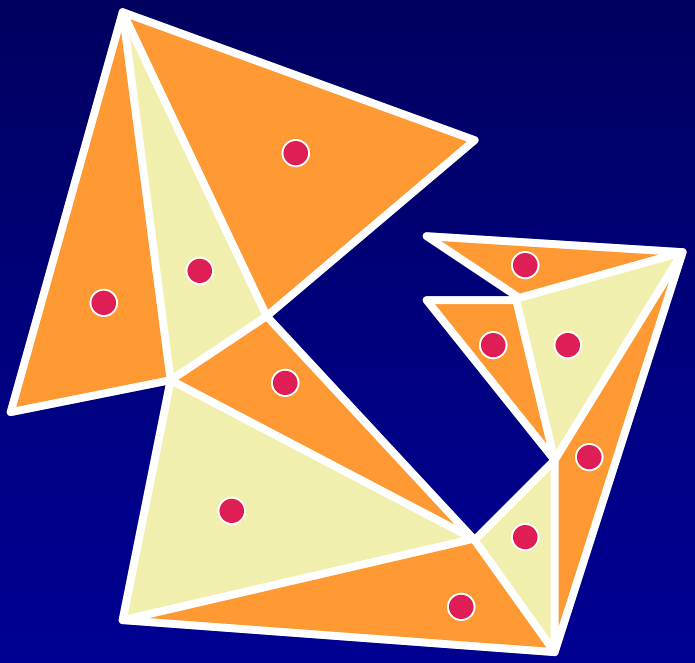
- The *entire* interior of a convex polygon is visible from *any* interior point. (**Why?**)

- A *star-shaped* polygon requires only one guard located in its *kernel*.



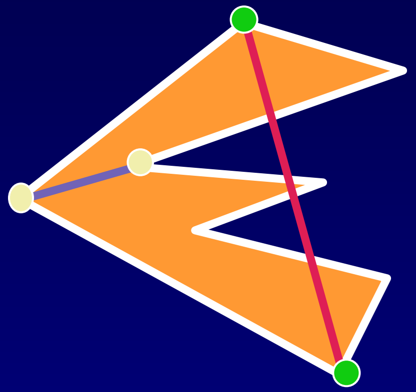
Art Gallery Problem: Easy Upper Bound

- **Theorem** (to be proven later): Every simple planar polygon with *n* vertices has a triangulation into *n*-2 triangles.
- *n*-2 guards suffice for an *n*-gon:
  - Subdivide the polygon into *n*-2 triangles (triangulation).
  - Place one guard in each triangle.

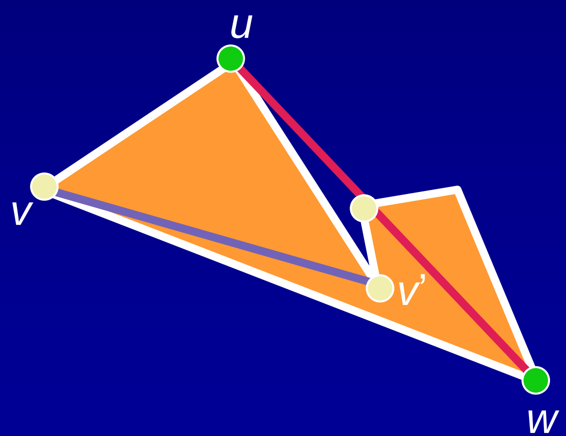


Diagonals in Polygons

- A *diagonal* of a polygon *P* is a line segment connecting two vertices, which lies **entirely** within *P*.



- **Theorem:** Every polygon with n>3 vertices has a diagonal.
- **Proof:** Find the leftmost vertex *v*. Connect its two neighbors *u* and *w*. If this is not a diagonal there must be other vertices inside the triangle *uvw.* Connect *v* to the vertex *v*' **farthest** from the segment *uw*. This must be a diagonal.



- **Questions:**
  1. Why is *v'v* a diagonal?
  2. Why not connect *v* with the leftmost vertex inside *uvw*?

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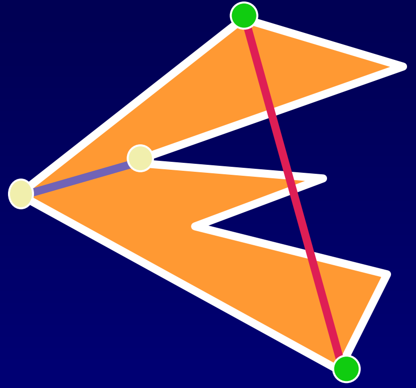

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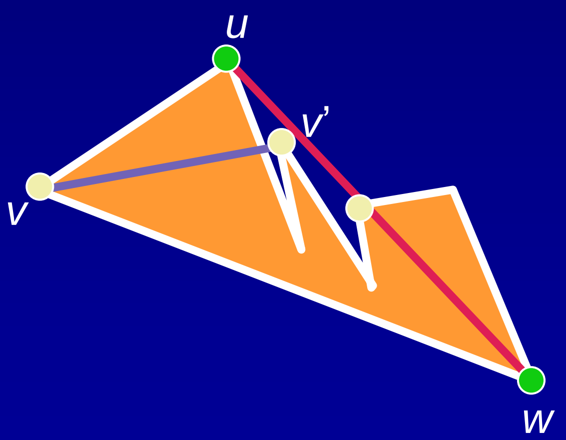
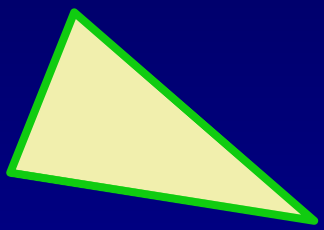


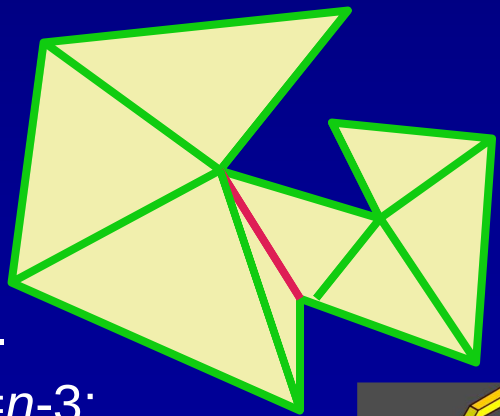
- **Questions:**
  1. Why is *v'v* a diagonal?
  2. Why not connect *v* with the leftmost vertex inside *uvw*?

Complexity of Triangulations

- **Theorem:** Any triangulation of a simple polygon with *n* vertices consists of *n*-3 diagonals and *n*-2 triangles.

- **Proof:** By induction on *n*:
  - Basis: A triangle (*n*=3) has a triangulation (itself) with no diagonals and one triangle.
  - Inductive step:
    1. For an *n*-vertex polygon, construct a diagonal dividing the polygon into two polygons with n_1 and n_2 vertices such that $n_1+n_2-2=n$. (Why "-2"?)
    2. Triangulate the two parts of the polygon.
    3. Diagonals: $(n_1-3)+(n_2-3)+1=(n_1+n_2-2)-3=n-3$;
       Triangles: $(n_1-2)+(n_2-2)=(n_1+n_2-2)-2=n-2$.





$\Theta(n^2)$-Time Polygon Triangulation

- Algorithm:
  1. Input: A simple n-gon.
  2. Find a diagonal.
  3. Call the algorithm recursively for the two subpolygons.

- Analysis: $$T(n) = O(n) + \max_{n_1+n_2=n+2} (T(n_1) + T(n_2))$$

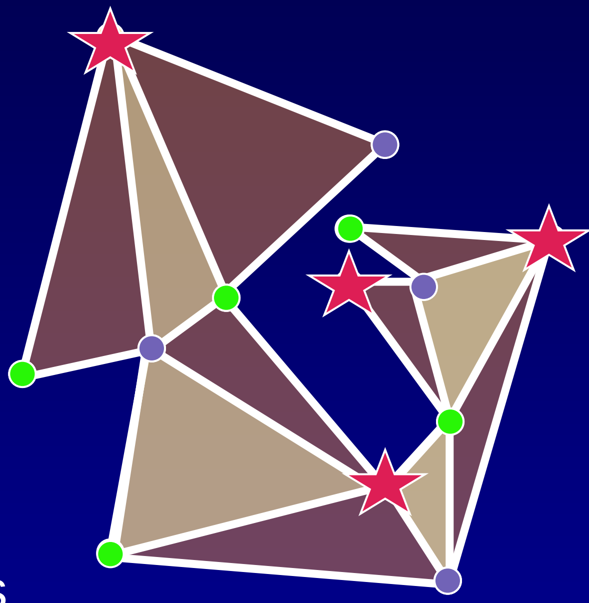
  diagonal recursion

  $$T(3), T(4) = \Theta(1)$$

- Solution: $T(n) = \Theta(n^2)$ time

- Space: $\Theta(n)$

Art Gallery Problem: Upper Bound



- Color the vertices of the (triangulated) polygon with three colors such that there is no edge between two vertices with the same color.
- **Question**: Why is this possible?

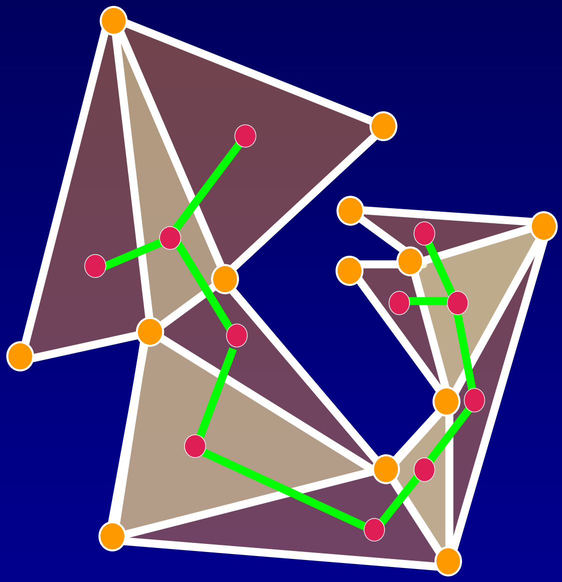
  (Hint: The dual of any triangulation is a tree with vertex degree at most 3. Full proof later.)
- Corollary: All triangles are 3-colored.
- Pick the color that is the least used. This color is used in at most $\lfloor n/3 \rfloor$ vertices.
- Place a guard on each vertex with this color. Due to the corollary all the triangles are guarded!
- $\Rightarrow$ New upper bound: $\lfloor n/3 \rfloor$

3-Coloring

- **Theorem:** Every triangulated polygon can be 3-colored.
- **Proof:** Consider the *dual* graph.
  - Since every diagonal disconnects the polygon, the dual graph is a **tree**.
  - Since every node in the graph is the dual of a triangle, its degree is ≤ 3.
  - Since any tree has a leaf, any triangulation has an ear (a triangle containing two polygon edges).
  - Finally, by induction on *n*:

    Basis: Trivial if *n*=3.

    Induction: Cut off an ear. 3-color the remaining (n-1)-gon. Color the *n*th vertex with the third color different from the two on its supporting edge.



3-Coloring

- **Theorem:** Every triangulated polygon can be 3-colored.
- **Proof:** Consider the *dual* graph.
  - Since every diagonal disconnects the polygon, the dual graph is a **tree**.
  - Since every node in the graph is the dual of a triangle, its degree is ≤ 3.
  - Since the polygon has *n* edges while any triangulation has *n*-2 triangles, any triangulation has an ear (a triangle containing two polygon edges).
  - Since any tree has at least one leaf, any triangulation has an ear (a triangle containing two polygon edges).
  - Finally, by induction on *n*:

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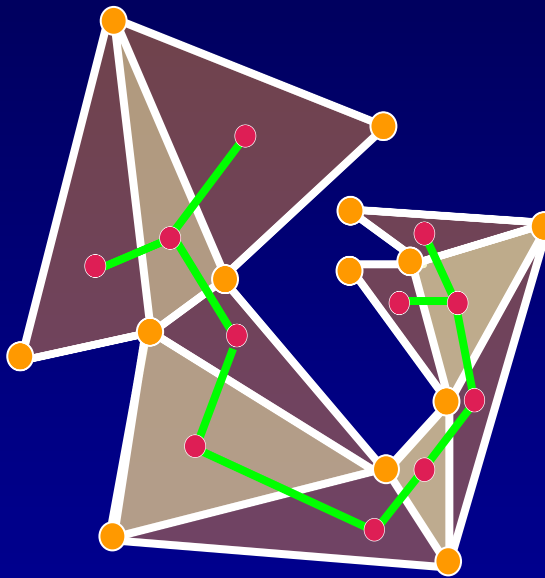
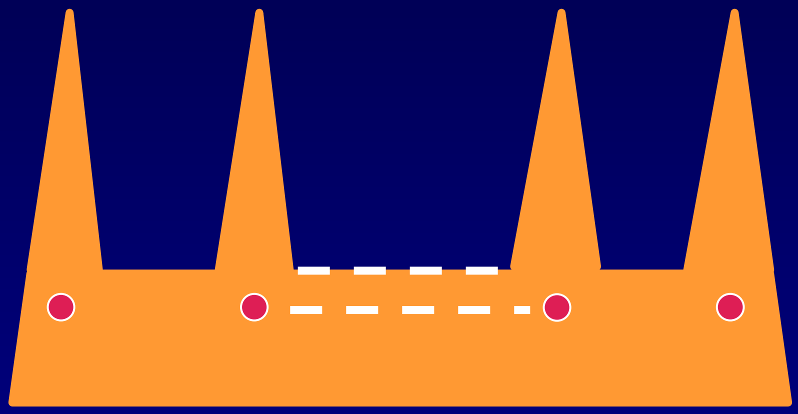
    Induction: Cut off an ear. 3-color the remaining (n-1)-gon. Color the *n*th vertex with the third color different from the two on its supporting edge.



A Matching Lower Bound

- Fact: There exists a polygon with *n* vertices, for which *n*/3 guards are necessary.



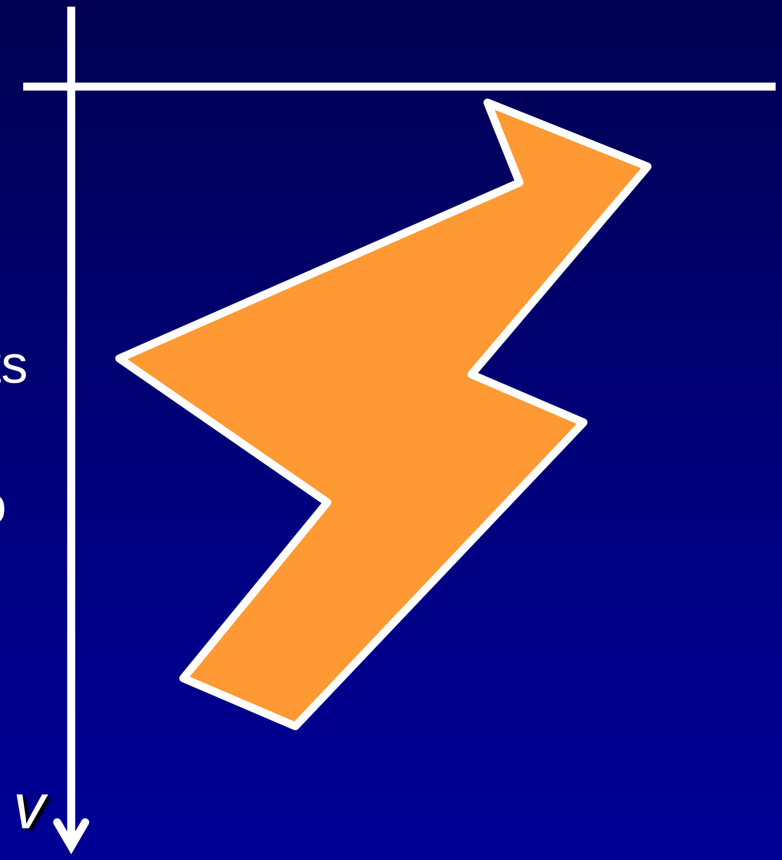
- Therefore, $\lfloor n/3 \rfloor$ guards are needed in the worst case.

O(n log n)-Time Polygon Triangulation

- A simple polygon is called *monotone* with respect to a direction v if for any line ℓ perpendicular to v, the intersection of the polygon with ℓ is connected.
- A polygon is called *monotone* if there exists any such direction v.
- A polygon that is monotone with respect to the x- (or y-) axis is called *x- (or y-) monotone.*

Question 1: How can we check in O(n) time whether a polygon is y-monotone?

Question 2: What is a polygon that is monotone with respect to **all** directions?

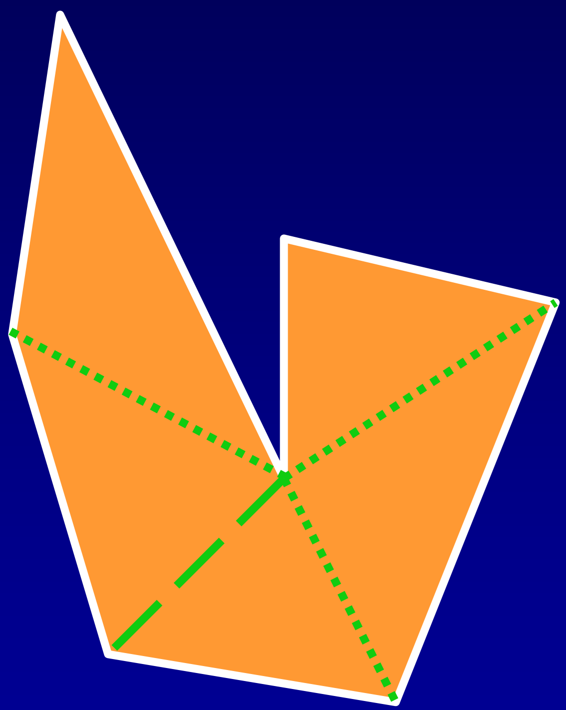


y-monotone but not *x*-monotone polygon

Triangulation Algorithm – cont.

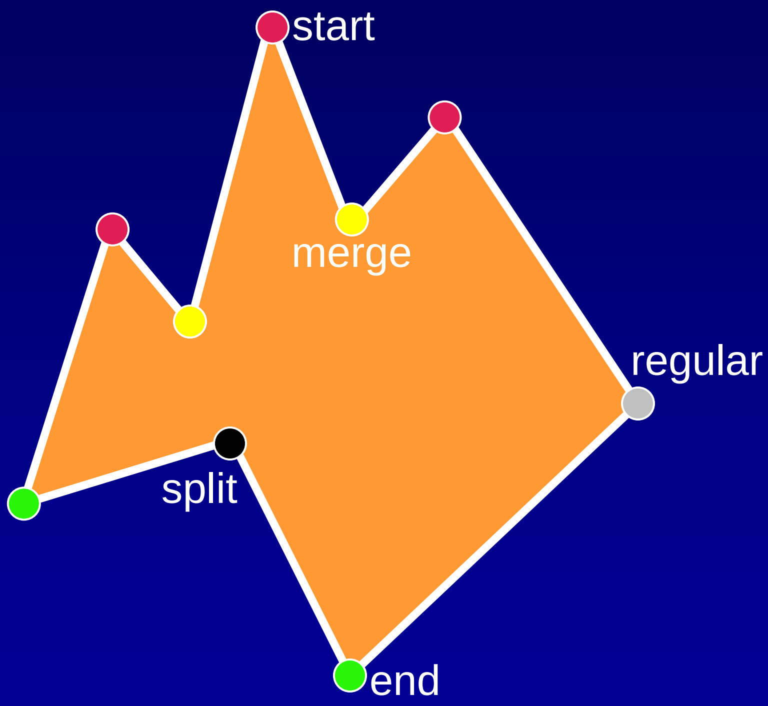
1) Partition the polygon into *y*-monotone pieces ("*חתיכות מונוטוניות*").

2) Triangulate each *y*-monotone piece separately.



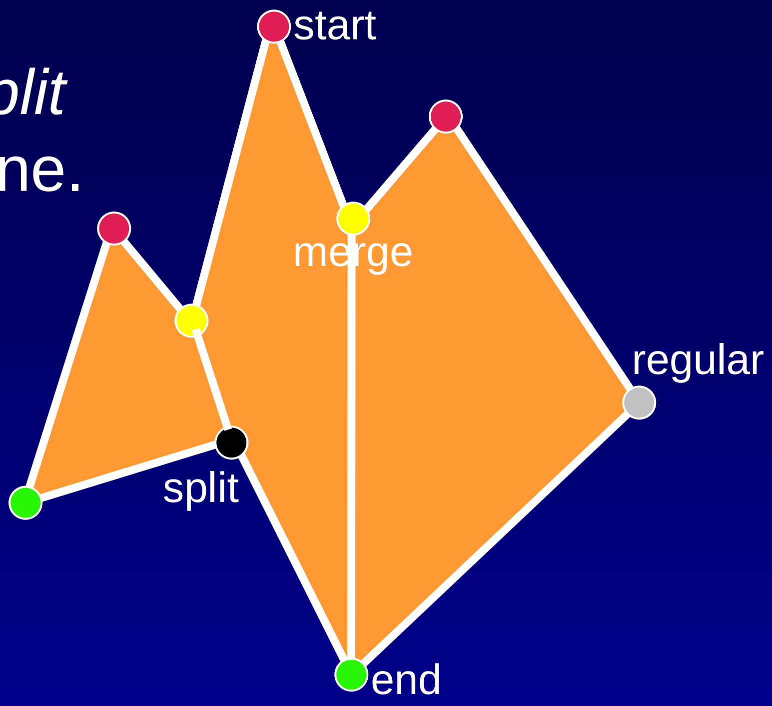
y-Monotone Polygons

- Classifying polygon vertices:
  - A *start* (resp., *end*) vertex is a vertex whose interior angle is less than π and its two neighboring vertices both lie below (resp., above) it.
  - A *split* (resp., *merge*) vertex is a vertex whose interior angle is greater than π and its two neighboring vertices both lie below (resp., above) it.
  - All other vertices are *regular*.



y-Monotone Polygons (cont.)

- **Theorem:** A polygon without *split* and *merge* vertices is *y*-monotone.

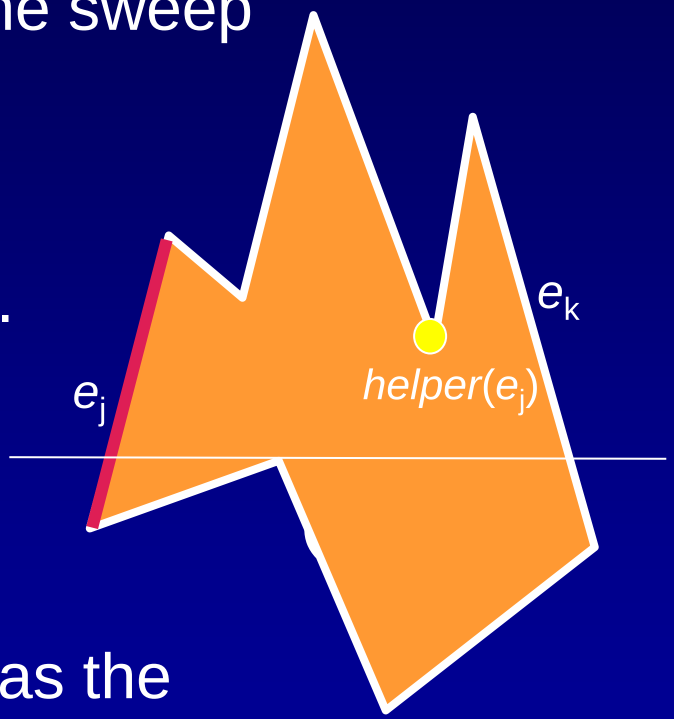


- Proof: Since there are only start/end/regular vertices, the polygon must consist of two *y*-monotone chains.

- To partition a polygon into monotone pieces, eliminate split (merge) vertices by adding diagonals upward (downward) from the vertex.
  Naturally, the diagonals **must not** intersect!

Monotone Partitioning

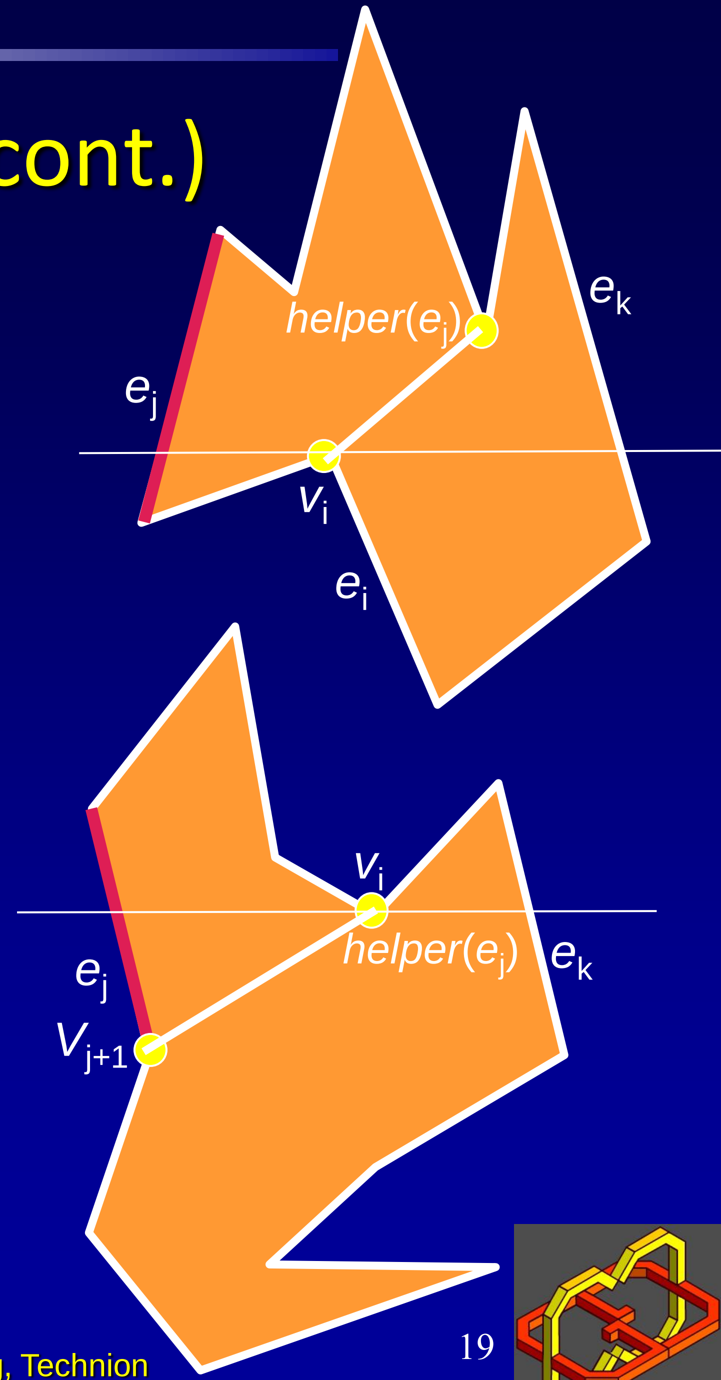
- Classify all vertices.
- Sweep the polygon from top to bottom.
- Maintain the edges intersected by the sweep line in a *sweep line status* (SLS sorted by *x* coordinates).
- Maintain vertex events in an event queue (EQ sorted by *y* coordinates). events are known in advance!
- Eliminate split/merge vertices by connecting them to other vertices be explained later).
- ***** For each edge *e*, define *helper*(*e*) as the lowest vertex (seen so far) above the sweep line, which is **visible** to the edge on its right side.
- *helper*(*e*) is initialized by the upper endpoint of *e*.



Monotone Partitioning (cont.)

- A split vertex may be connected to the helper vertex of the edge immediately to its left.
- However, a merge vertex should be connected to a vertex which has not been processed yet!
- Clever idea: Every merge vertex *v* is the helper of some edge *e*, so that *v* will be resolved either
  - when *e* disappears; or
  - when *v* ceases to be the helper of *e*.

It will be the last time *v* can be resolved!



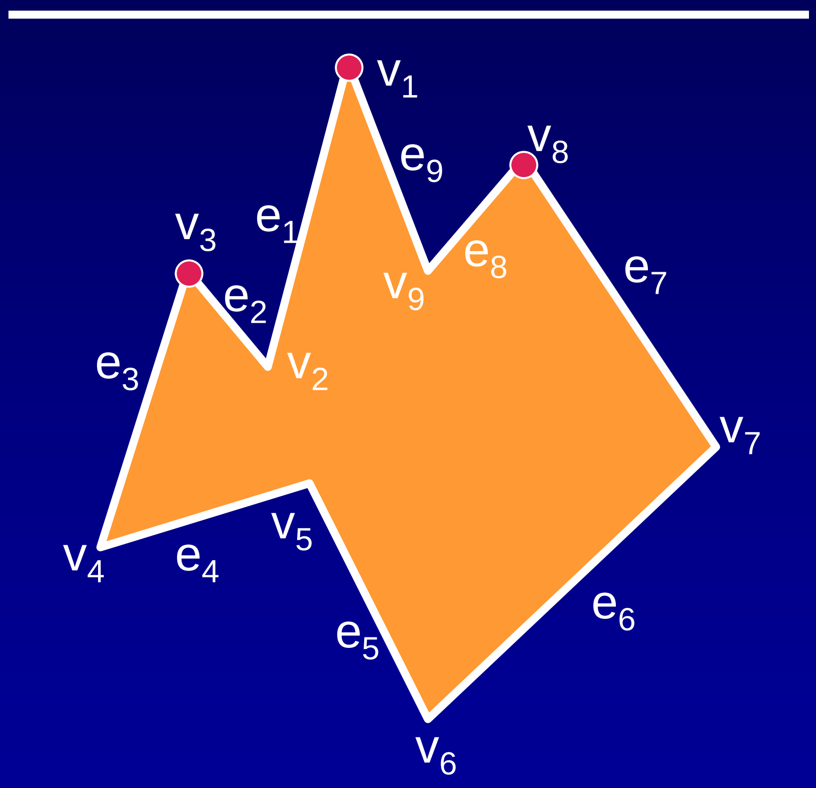
Monotone Partitioning Algorithm

- Input: A polygon *P*, given as a list of vertices ordered counterclockwise. The edge e_i immediately follows the vertex v_i.
- Construct EQ containing the vertices of *P* sorted by their *y*-coordinates. (In case two or more vertices have the same *y*-coordinate, the vertex with the smaller *x*-coordinate has a higher priority.)
- Initialize SLS to be empty.
- While EQ is not empty:
  - Pop vertex *v*;
  - Handle *v*.

  (No new events are generated during execution.)
- Idea: No split/merge vertex remains unhandled!

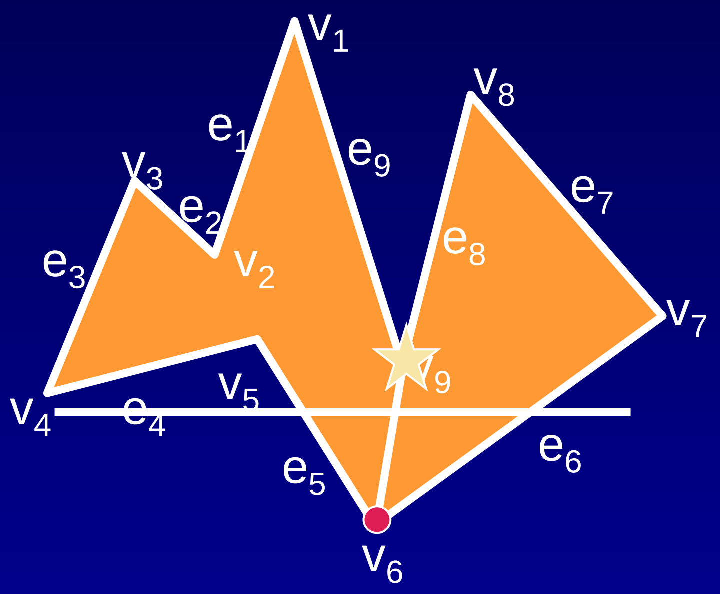
Start Vertices

- Handling a *start* vertex (v_i):
  - Add e_i to SLS
  - $helper(e_i) := v_i$
- Implementation detail: Only “left” edges (for which the polygon is on the right) need a helper and are thus kept in the status.



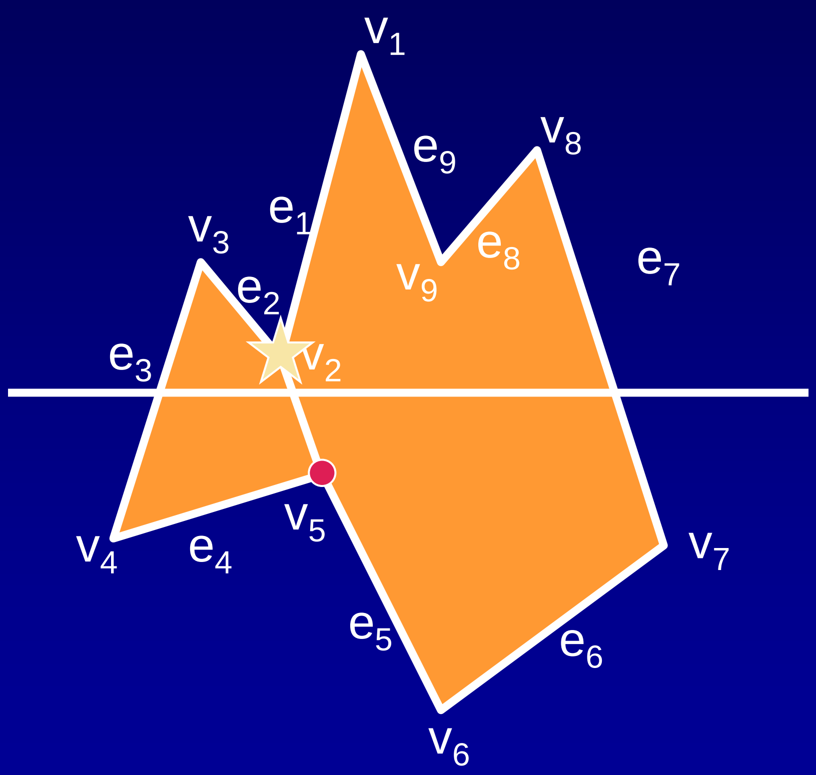
End Vertices

- Handling an *end* vertex (v_i):
  - If *helper*(e_{i-1}) is a merge vertex, then connect v_i to *helper*(e_{i-1}) (Why?!)
  - Remove e_{i-1} from SLS



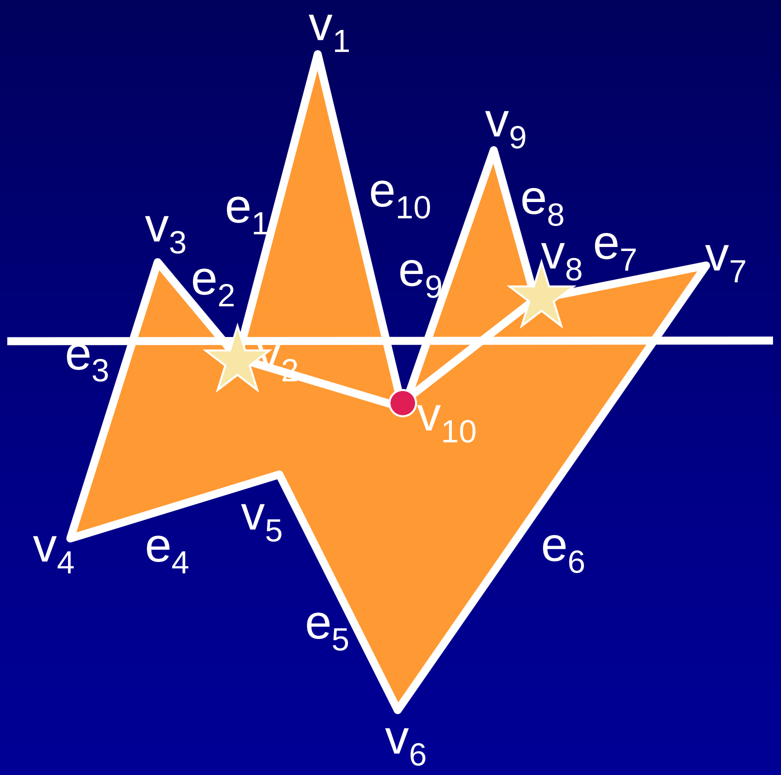
Split Vertices

- Handling a *split* vertex (v_i):
  - Find in SLS the edge e_j directly to the left of v_i
  - Connect v_i to *helper*(e_j)
  - *helper*(e_j) := v_i
  - Insert e_i into SLS
  - *helper*(e_i) := v_i



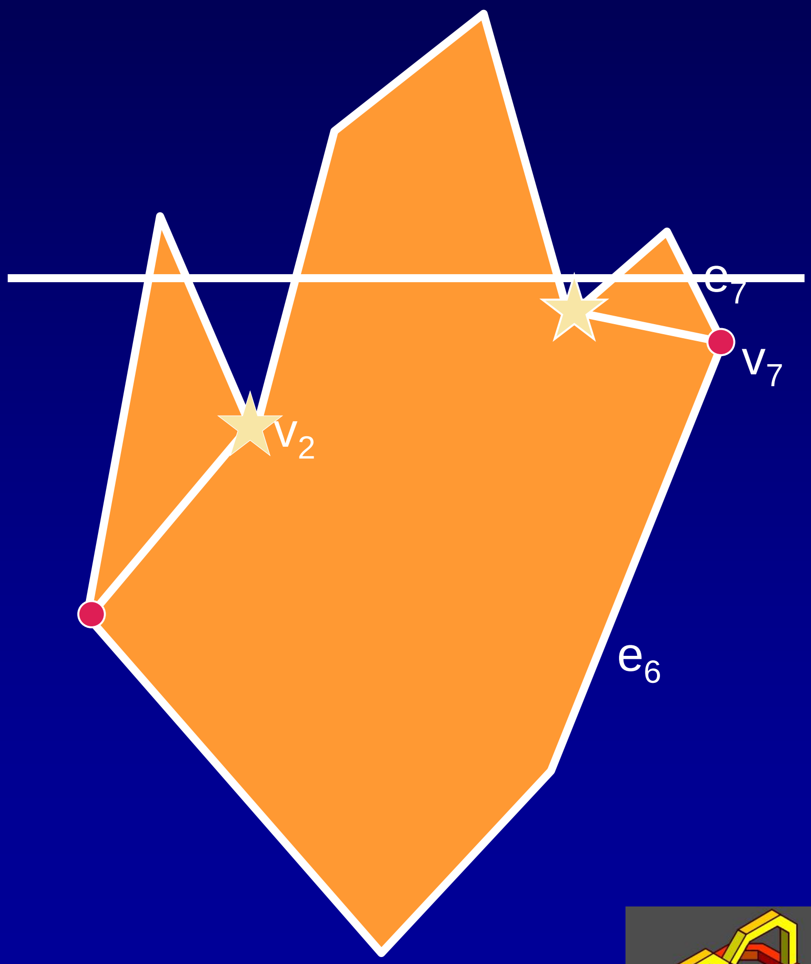
Merge Vertices

- Handling a *merge* vertex (v_i):
  - If *helper*(e_{i-1}) is a merge vertex, then connect v_i to *helper*(e_{i-1})
  - Remove e_{i-1} from SLS
  - Find in SLS the edge e_j directly to the left of v_i
  - If *helper*(e_j) is a merge vertex, then connect v_i to *helper*(e_j)
  - *helper*(e_j) := v_i



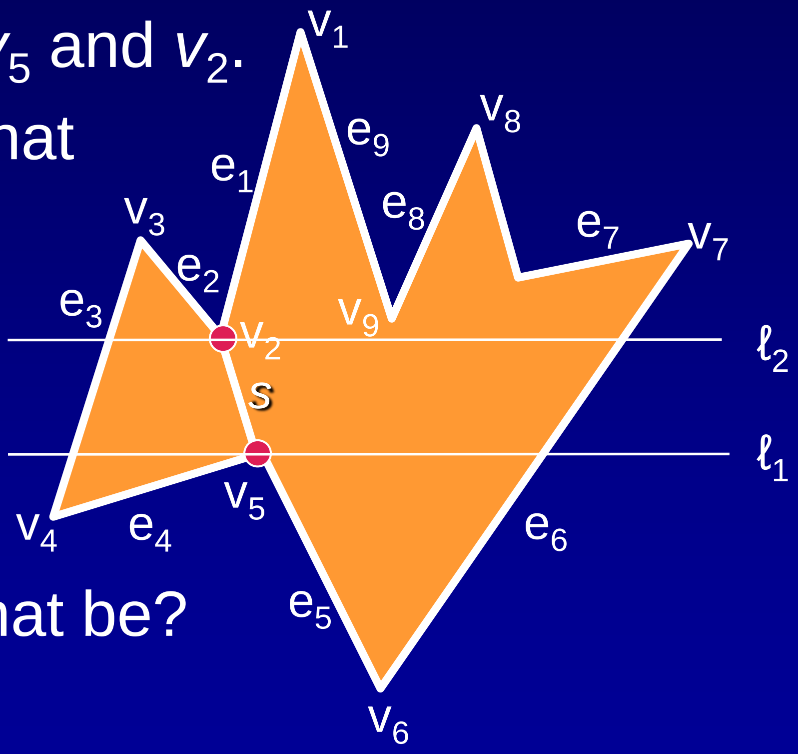
Regular Vertices

- Handling a *regular* vertex (v_i):
  - If the polygon's interior lies to the left of v_i then:
    - Find in SLS the edge e_j directly to the left of v_i
    - If *helper*(e_j) is a merge vertex, then connect v_i to *helper*(e_j)
    - *helper*(e_j) := v_i
  - Else:
    - If *helper*(e_{i-1}) is a merge vertex, then connect v_i to *helper*(e_{i-1})
    - Remove e_{i-1} from SLS
    - Insert e_i into SLS
    - *helper*(e_i) := v_i



Proof of Correctness: Split Vertices

- Assume that the split vertex v_5 was connected to v_2.
- Assume that $s=v_5v_2$ intersects another original edge e.
- Draw horizontal lines through v_5 and v_2.
- Where can the endpoint of e, that is to the left of s, be?
  - Below ℓ_1: Impossible. (Why?)
  - Between ℓ_1 and ℓ_2: Ditto. (Why?)
  - Above ℓ_2: Ditto. (Why?)
- Now assume that s intersects another diagonal. Why can't that be?
- Conclusion:
  Split events are resolved correctly.



Proof of Correctness (cont.)

- Merge vertices: Exercise.

- Complete the details of the proof as an exercise.

Triangulating a *y*-monotone Polygon

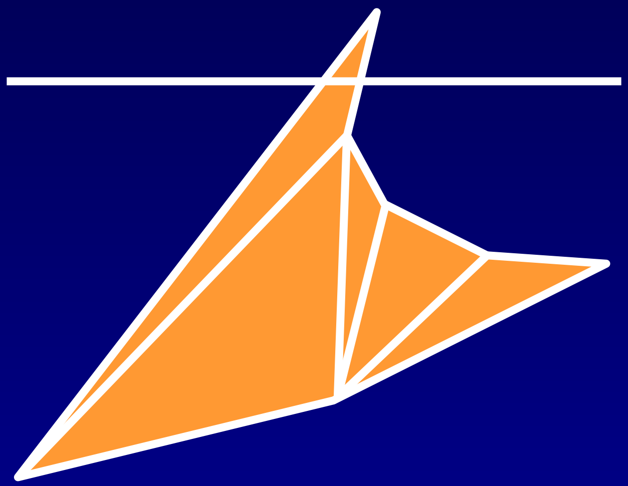
In Theory

- Sweep the polygon from top to bottom.
- Greedily triangulate anything possible above the sweep line, and then forget about this region.
  - When we process a vertex *v*, the unhandled region above it always has a simple structure: Two *y*-monotone (left and right) chains, each containing at least one edge. If a chain consists of two or more edges, it is *reflex*, and the other chain consists of a single edge whose bottom endpoint has not been handled yet.
  - Each diagonal is added in O(1) time.

Triangulating a Y-monotone Polygon

In Practice

- Continue sweeping while one chain contains only one edge, while the other edge is *concave*.
- When a "convex edge" appears in the concave chain, triangulate as much as possible by connecting the new vertices to all visible vertices of the concave chain.



- When the edge in the other chain terminates, connect it to all the vertices of the concave chain using a "fan".
- Time complexity: O(*k*), where *k* is the complexity of the polygon.

Question: Why?!

Total Time-Complexity Analysis

- Partitioning the polygon into monotone pieces:

  O(n log n)

  (every vertex event is handled in O(log n) time)

- Triangulating all the monotone pieces: Θ(n)

  (every vertex event is handled in Θ(1) time)

Total: O(n log n)

Historical Perspective

- $O(n^2)$: Diagonal insertion
- $O(n \log n)$: Lee and Preparata (Monotone decomposition, 1977)
  Avis and Toussaint (1981)
  Chazelle (1982)

  Optimal??

- $O(n \log \log n)$: Tarjan and Van Wyk (1988)
- $O(n \log^* n)$: **Randomized**:
  Clarkson, Tarjan, and Van Wyk (1989)
  Seidel (Trapezoidal decomposition, 1991)
  Devillers (1992)
- $\Theta(n)$: **Optimal** (yet deterministic):
  Chazelle (1991)