Computational Geometry

Chapter 5

Orthogonal Range Searching

On the Agenda

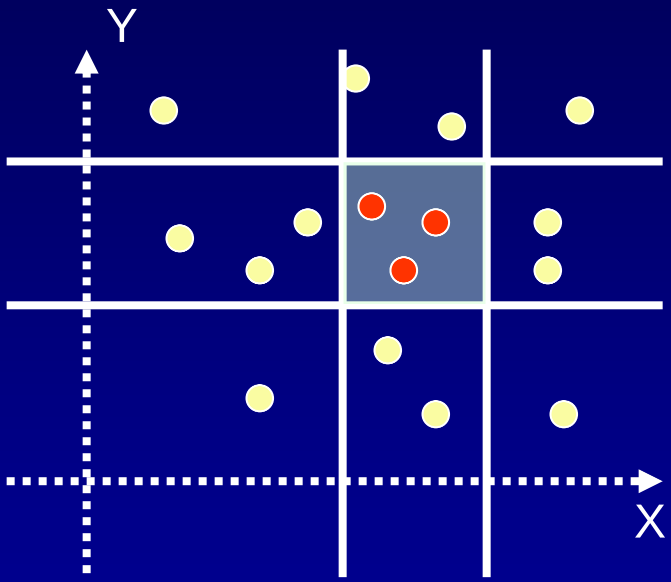
- *k*-D Trees
- Range Trees

Orthogonal Range Searching

- **Problem:** Given a set of *n* points in $\Re^d$, preprocess them such that reporting or counting the *k* points inside a *d*-dimensional axis-parallel box will be efficient.

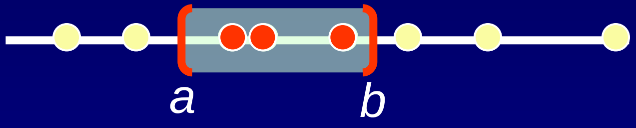
- Desired *output-sensitive* query time complexity – O(*k*+f(*n*)) for reporting and O(f(*n*)) for counting, where f(*n*)=o(*n*), e.g., f(*n*)=O(log *n*).

- **Sample application:** Report all cities within 20 KM radius of Tel Aviv. (Here the range is actually a circle.)



Range Searching: 1D

- In a one-dimensional space, points are real numbers, and a range is defined by two numbers *a* and *b*.

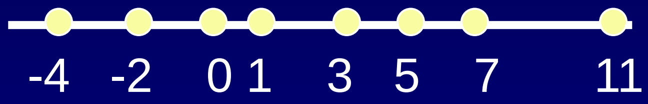


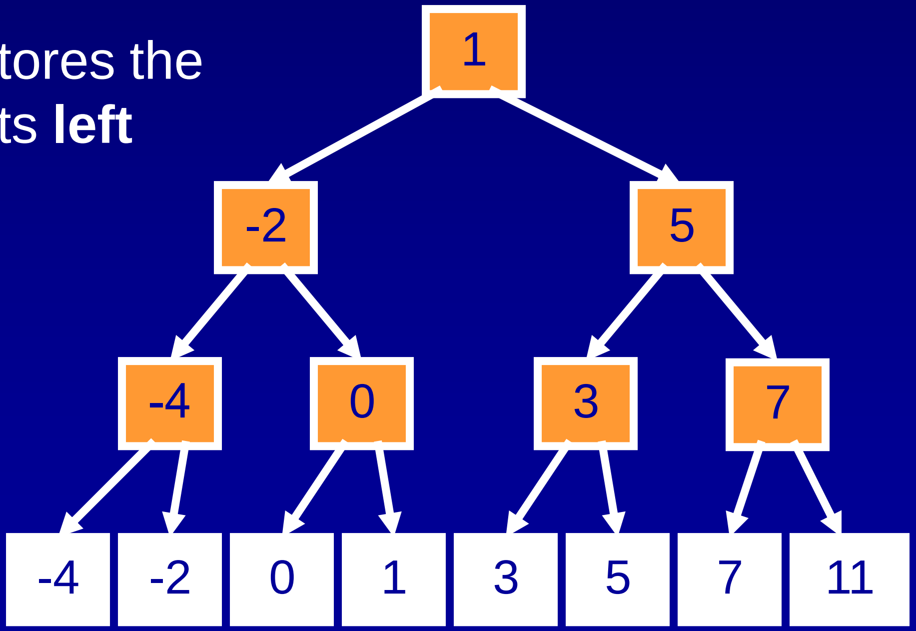
- A simple O(log *n*)-time algorithm:
  - Sort points (O(*n* log *n*) time preprocessing).
  - (Binary) search for *a* and *b* in the list (O(log *n*) time).
  - List all values in between.

- Cannot be easily generalized to higher dimensions. (Why not ?).

Range Searching: 1D Tree

- Range tree solution:
  - Sort points.
  - Construct a balanced binary tree, storing the points in its leaves.
  - Each tree node stores the **largest** value of its **left** subtree.



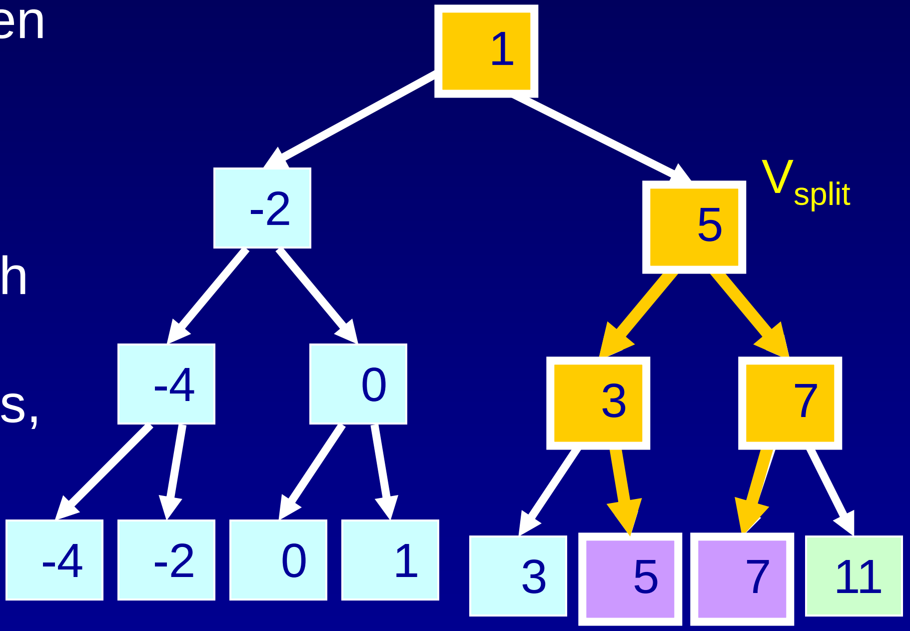


Range Searching in a 1D Tree

- Finding a leaf: O(log *n*) time.
- Find the two boundaries of the given range in the leaves *u* and *v*.
- Report all the leaves in *maximal subtrees* between *u* and *v*.
- Mark the vertex at which the search paths diverge as V_{split}.
- Continue to find the two boundaries, reporting values in the subtrees:

  When going towards the left (right) endpoint of the range: If going left (right), report the entire right (left) subtree.
- When reaching a leaf, it needs to be checked.

Input Range: 3.5-8.2



Running-Time Analysis

- k: output size
- Leaves: $O(k)$ time
- Internal nodes: $O(k)$ time (since this is a binary tree)
- Paths: $O(\log n)$ time
- Total: $O(\log n + k)$ time

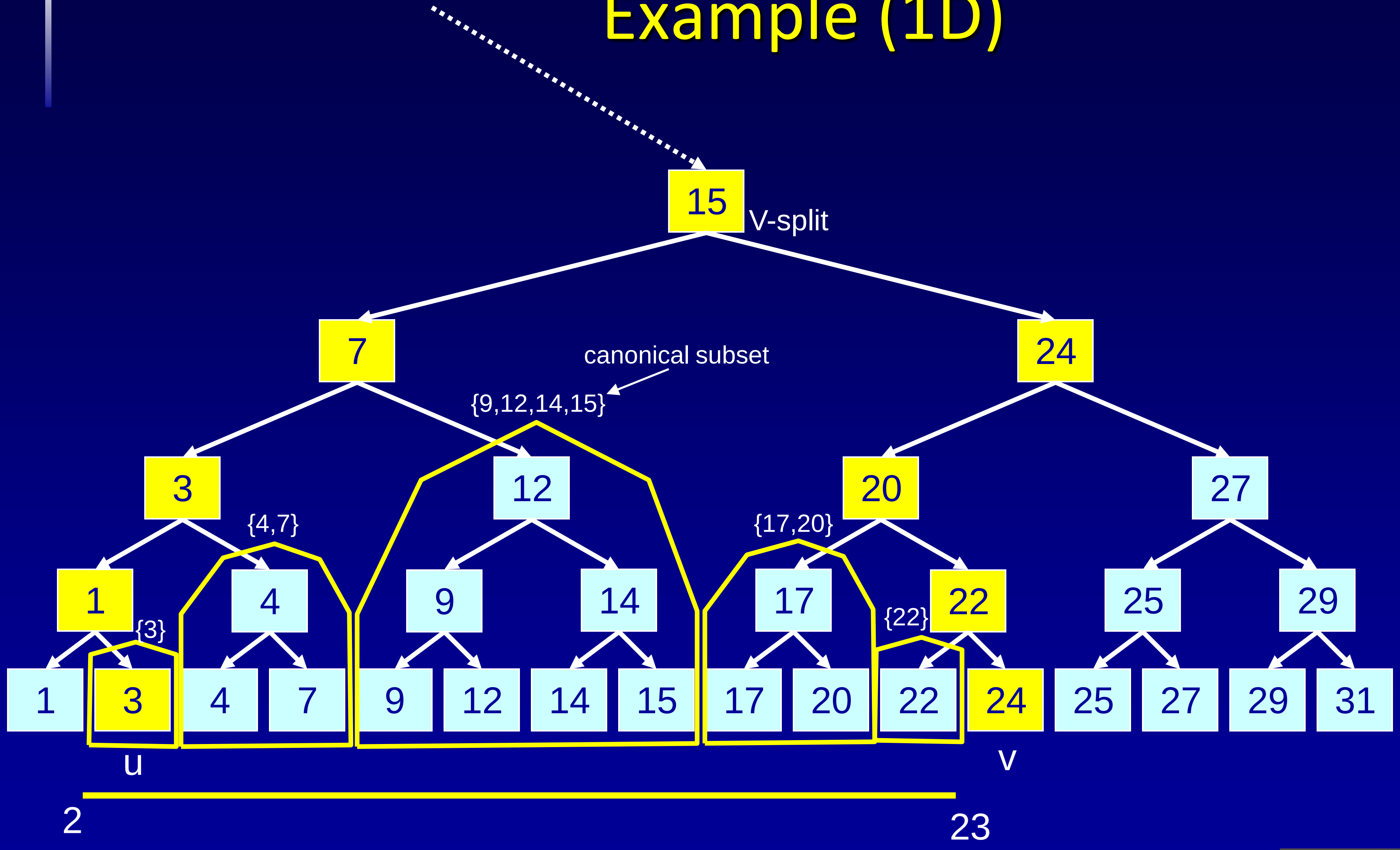
- Worst case: $k = n \rightarrow \Theta(n)$ time
- Counting: $O(\log n)$ even in the worst case. How?

General Idea

- Build a data structure storing a “small” number of canonical subsets, such that:
  - The canonical sets may overlap.
  - Every query may be answered as the union of a “small” number of canonical sets.
- Needs the geometry of the space to enable this.

- Two extremes:
  - Singletons: O(k) query time, even for counting.
  - Power set: O(1) query time, O(2^n) storage.

Example (1D)

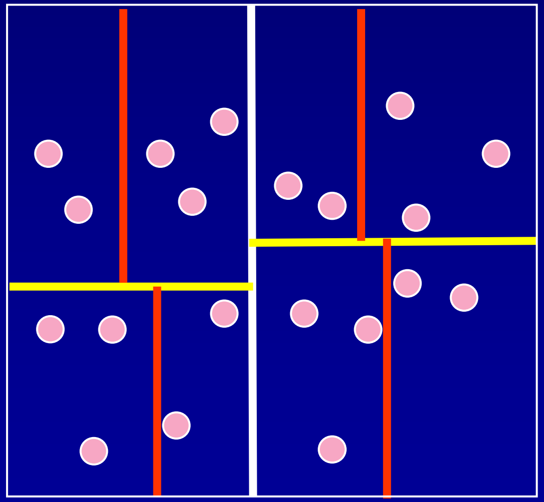


The canonical subsets are subtrees (overkill in 1D).

What is the space consumption?

2D Trees

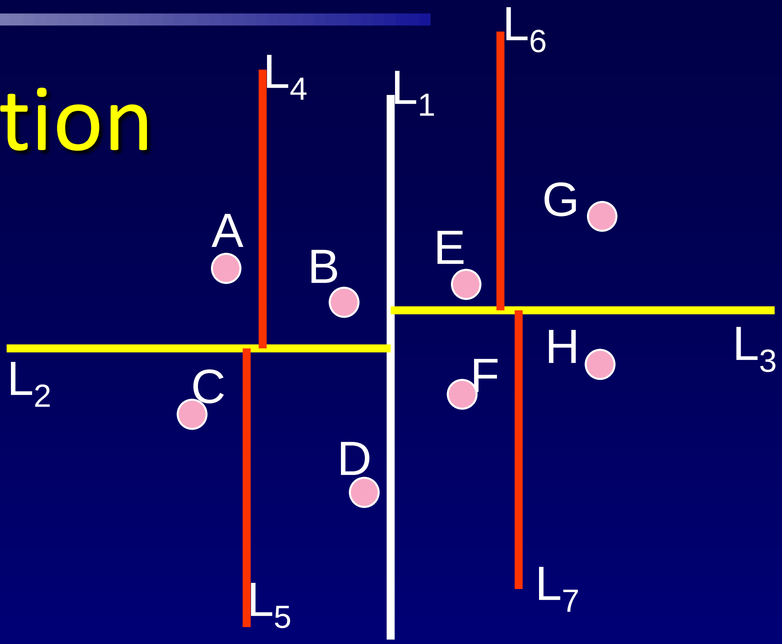
- Input: A set of points in 2D.
- Enclose the points by an axis-parallel rectangle.
- Split the points into two equal-size subsets, using a horizontal or vertical line.
- Continue recursively to partition the subsets, alternating the directions of the lines, until point subsets are small enough (of constant size).
- Canonical subsets are subtrees.



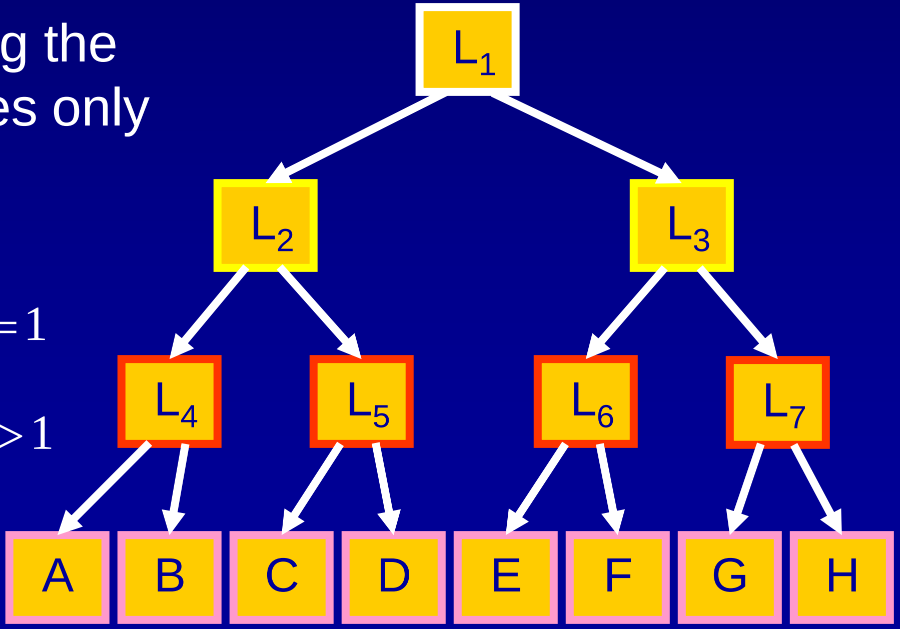
- In higher (*k*) dimensions: Split directions alternate between the *k* axes.
- In *k*-D it is called “*k*-D tree”.

  In 2-D: Used to be called “2-D tree”; now (slang) called “2-D *k*-D tree”.

2D Tree: Construction



- Partition the plane into axis-aligned rectangular regions.
- Nodes represent rectangles (and partition lines), and leaves represent input points.



- The bottleneck is finding the median, but this requires only linear time!
- Time complexity:

$$T(n) = \begin{cases} O(1) & n = 1 \\ O(n) + 2T\left(\frac{n}{2}\right) & n > 1 \end{cases}$$

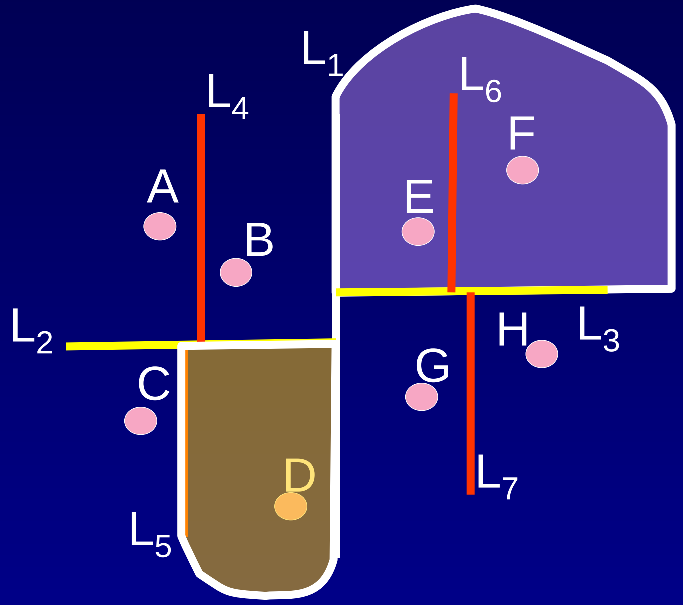
$$T(n) = O(n \log n)$$

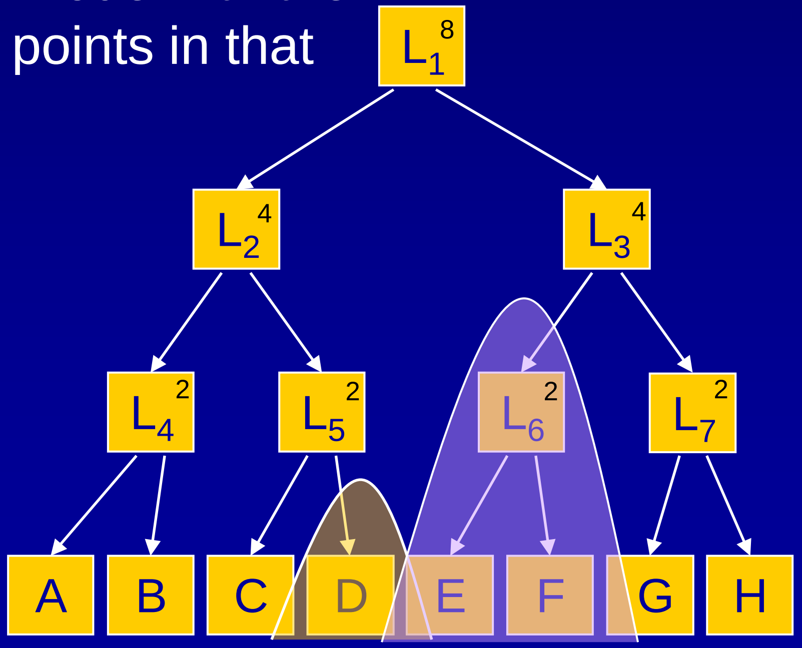
Two Possible Improvements

- Instead of finding the median from scratch each time:
  - Spend (twice) O(*n* log *n*) preprocessing time on sorting all points (once according to x, and once according to y).
  - Finding the median will be easier, but will still require linear time.
- Questions:
  - Why linear and not, say, logarithmic time?
  - Is it an asymptotic improvement?
- Attempting to overcome the last pitfall, copy the point subsets to the children trees (to avoid “jumps”). Thus, finding the median will require **constant** time. Unfortunately asymptotically there will be no improvement. Why?

Range Counting/Reporting

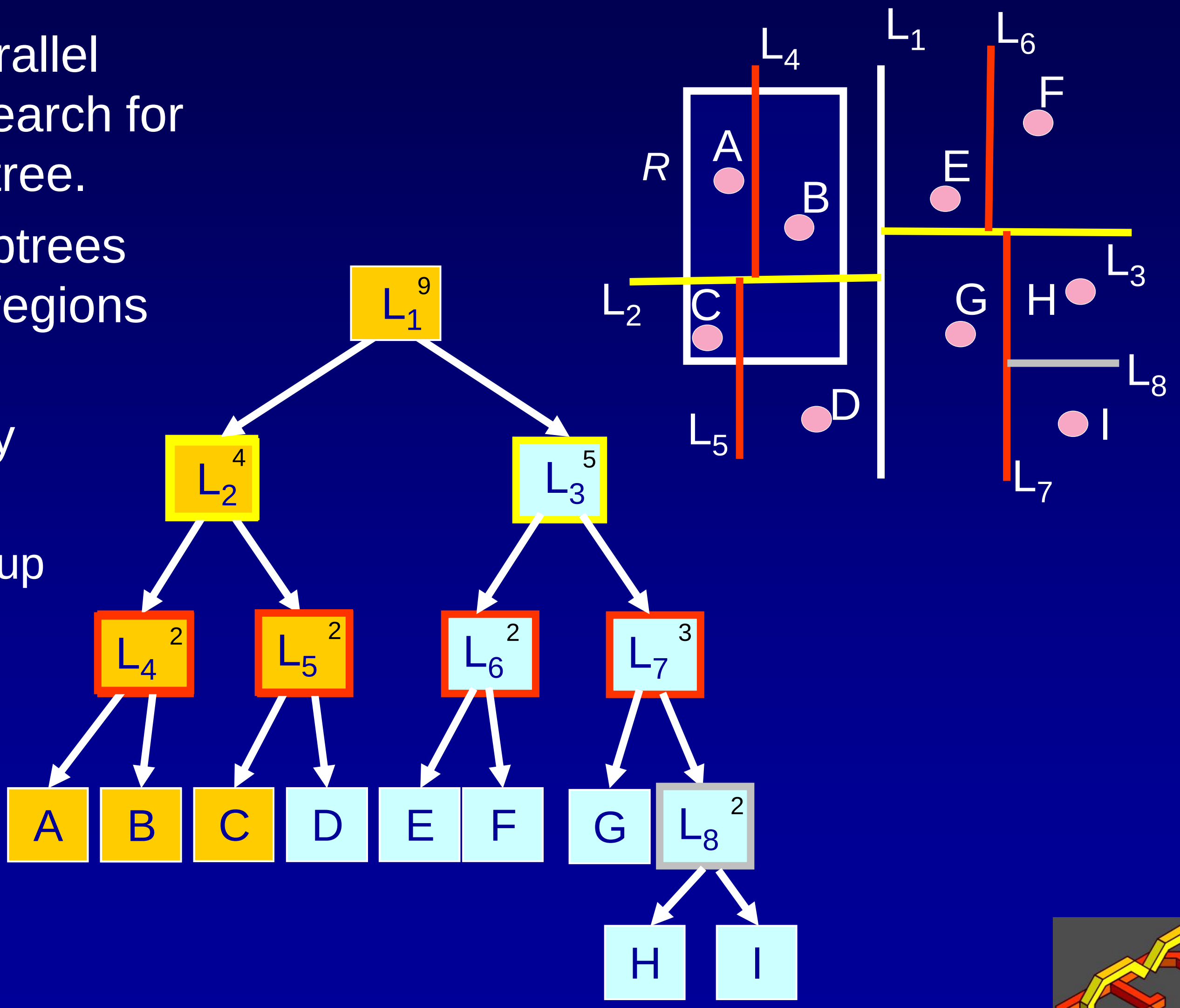
- Each node in the tree defines an axis-parallel rectangle in the plane, bounded by the lines marked by this vertex's ancestors.
- Label each node with the number of points in that rectangle.





Range Counting/Reporting (cont.)

- Given an axis-parallel range query *R*, search for this range in the tree.
- Traverse only subtrees which represent regions **overlapping** *R*.
- If a subtree entirly contained in R:
  - Counting: Add up its count.
  - Reporting: Report entire subtree.



Time-Complexity Analysis

- *k* nodes are reported. How much time is spent on internal nodes? The nodes visited are those that are **stabbed** by *R* but are not contained in *R*. How many such nodes exist?
- **Theorem**: Every side of *R* stabs O(√*n*) cells of the tree.
- **Proof**: Extend the side (w.l.o.g., horizontal) to a full line.
  In the first level it stabs two children, and in the next level it stabs two out of the four grandchildren.
  By the Master Theorem, $Q(n) = O(\sqrt{n})$.

  $$Q(n) = \begin{cases} 1 & n = 1 \\ 2 + 2Q\left(\dfrac{n}{4}\right) & \text{else} \end{cases}$$
- Total query time: $O(\sqrt{n} + k)$.

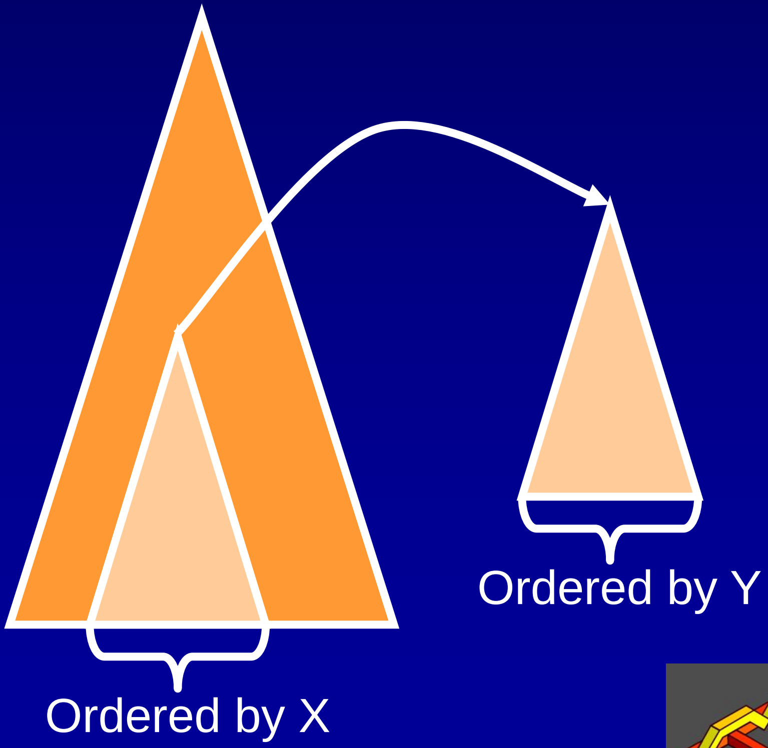
*k*d-Trees: Higher Dimensions

- For a d-dimensional space:
  - Same algorithm
  - O(d) time is needed to handle a *single* point
  - Construction time: O($d\ n \log n$)
  - Space Complexity: O($d\ n$)
  - Query time complexity: O($d\ (n^{1-1/d}+k)$)
- Note: For large d, full scan is almost equally good!
- **Question:** Are *k*d-trees useful for non-orthogonal range queries, e.g., disks, convex polygons?

- Compare with the performance of d-D range trees (at the end of this presentation)

Multi-Level Data Structure

- Construct a tree ordered by *x* coordinates.
- Each inner vertex *v* contains a pointer to a secondary tree, that contains all the points of the primary subtree ordered by *y* coordinates.
- Points are stored **only** in the secondary trees.



Range Tree: Construction

- Same as a 1D-Tree, except that in each level the secondary trees are built as well.
- **Theorem**: The space complexity is Θ(*n* log *n*).
- **Proof**: The size of the primary tree is Θ(*n*). Each of its Θ(log *n*) levels corresponds to a collection of secondary trees that contains **all** the *n* points.
- Construction time (naïve analysis):

$$T(n) = \begin{cases} O(1) & n = 1 \\ O(n \log n) + 2T\left(\frac{n}{2}\right) & \text{else} \end{cases}$$

$$= O(n \log^2 n)$$

Range Tree: Improved Construction

- However, there is no need for **repeated** sorting by *y* coordinates!
- Instead, we can sort by *y* only **once** (in O(*n* log *n*) time), and copy data in the recursive calls in linear time.
- The resulting recursive equation is:

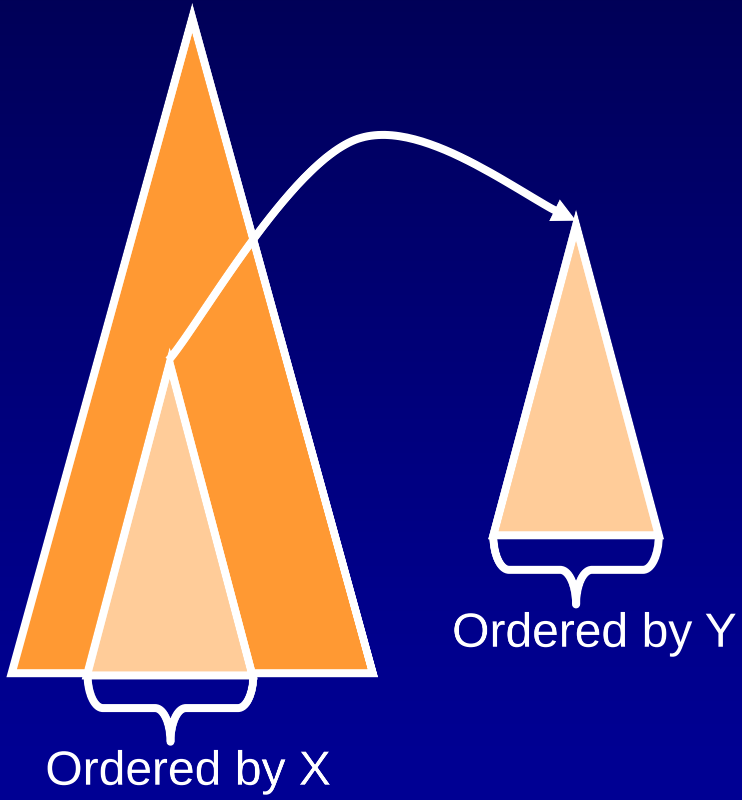
$$T(n) = \begin{cases} \mathrm{O}(1) & n = 1 \\ \mathrm{O}(n) + 2T\left(\frac{n}{2}\right) & \text{else} \end{cases}$$

$$= \mathrm{O}(n \log n)$$

- Overall: O(*n* log *n*) time.

Range Tree: Search

- Given a 2D range, we simulate a 1D search and find subtrees sorted by *x*.
- Instead of reporting the entire subtrees, we *filter* them by invoking a search in the secondary trees sorted by *y*, and report only the points in the query range.



Search: Analysis

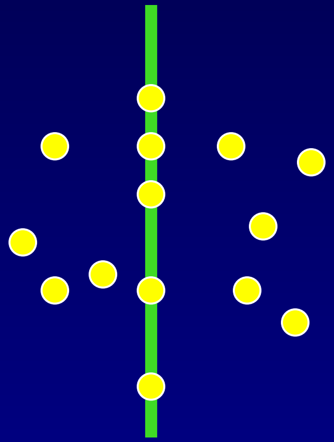
- Time complexity:

$$T(n) = \underbrace{O(\log n)}_{\text{traversing primary structure}} + \underbrace{\sum_v}_{\text{calls to secondary structure}} (\underbrace{\log n}_{\text{traversing secondary structure}} + \underbrace{k_v}_{\text{reporting}}) = O(\log^2 n + k)$$

- The running time can be reduced to O(log *n* + *k*) by using *fractional cascading*.

Points in Non-General Position

- Question: How can we handle sets of points which are not in general position, i.e., with multiple points with the same *x* coordinate?
- Answer: By two-step order checks. When comparing according to *x*, resolve ties by *y*, and vice versa.
- This splits points into two sides, having the same effect as infinitesimally rotating the plane.
- **Theorem**: The modified order checks preserve the correctness of the algorithms.



Range Trees: Higher Dimensions

- Preprocessing: O($d\ n \log^{d-1} n$) time
- Space: O($d\ n \log^{d-1} n$)
- Query: O($d\ (\log^{d-1} n + k)$) time