Computational Geometry

Chapter 8

Duality

Center for Graphics and Geometric Computing, Technion

On the Agenda

- Order-preserving duality
- Non-order-preserving dualities

Order-Preserving Duality

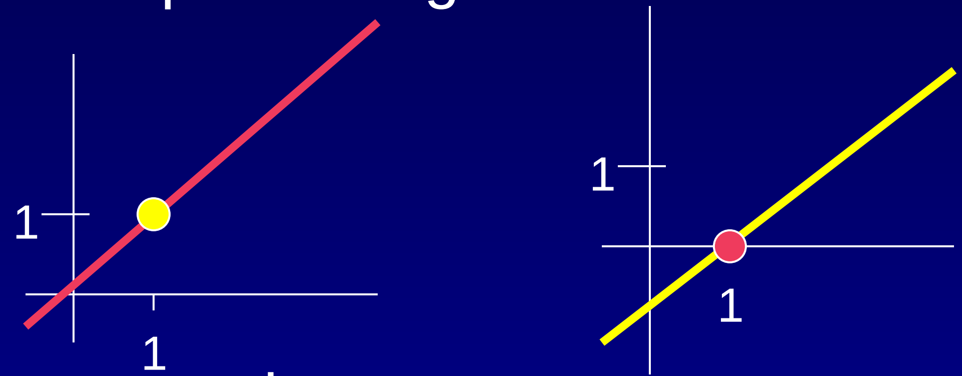
Point: $P(a,b)$	Dual line: $P^*: y=ax-b$
Line: $\ell : y=ax+b$	Dual point: $\ell^*: (a,-b)$

Note: Vertical lines (x=C, for a constant C) are not mapped by this duality (or, actually, are mapped to "points at infinity"). We ignore such lines since we can:

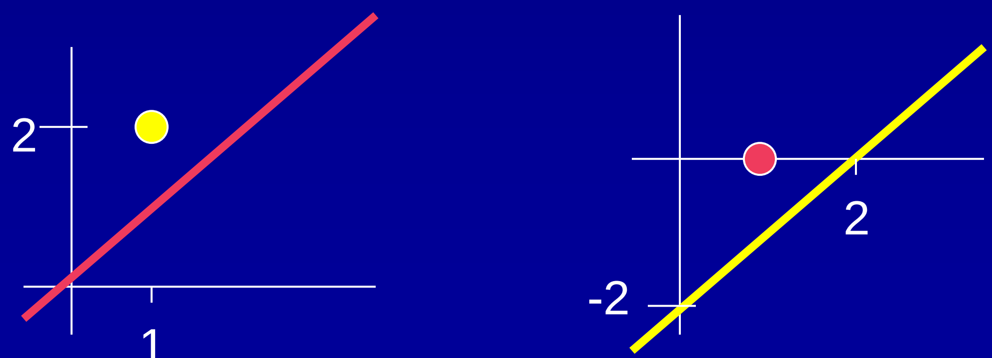
- Avoid vertical lines by a slight rotation of the plane; or
- Handle vertical lines separately.

Duality Properties

1. Self-inverse: $(P^*)^* = P$, $(\ell^*)^* = \ell$.
2. Incidence preserving: $P \in \ell \Leftrightarrow \ell^* \in P^*$.



3. Order preserving:
   P above/on/below $\ell \Leftrightarrow \ell^*$ above/on/below P^*
   (the point is always below/on/above the line).

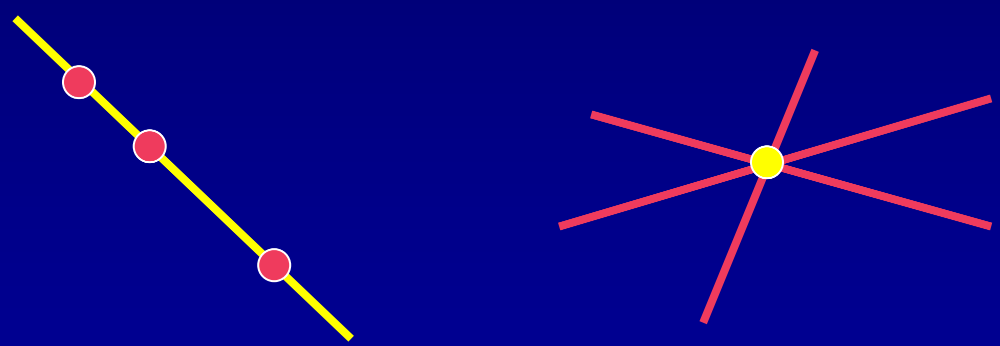


Duality Properties (cont.)

4. Points P_1, P_2, P_3 collinear on ℓ

$$\Updownarrow$$

Lines P_1^*, P_2^*, P_3^* intersect at ℓ^*.

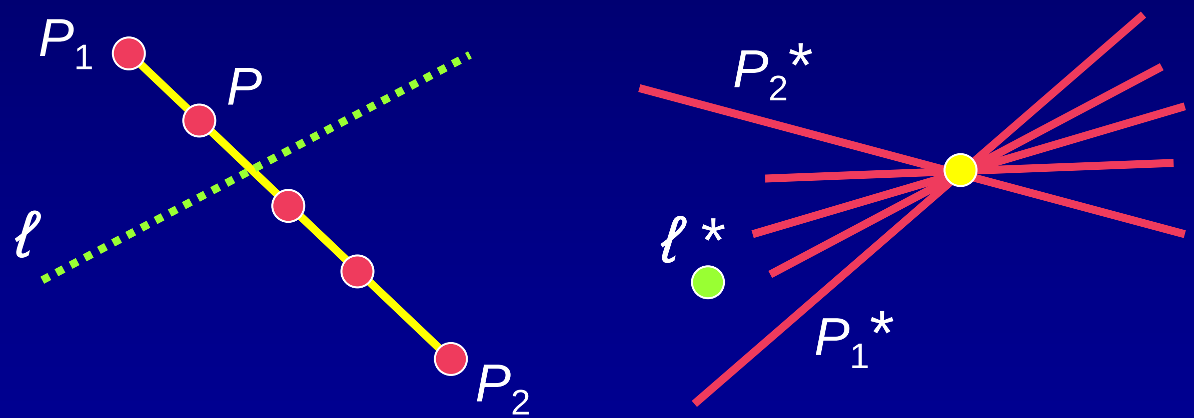


(Follows directly from property 2.)

Duality Properties (cont.)

5. The dual of a line segment $s=[P_1P_2]$ is a *double wedge* that contains all the dual lines of points P on s.

   All these points P are collinear, therefore, all their dual lines intersect at one point, the intersection of P_1^* and P_2^*.

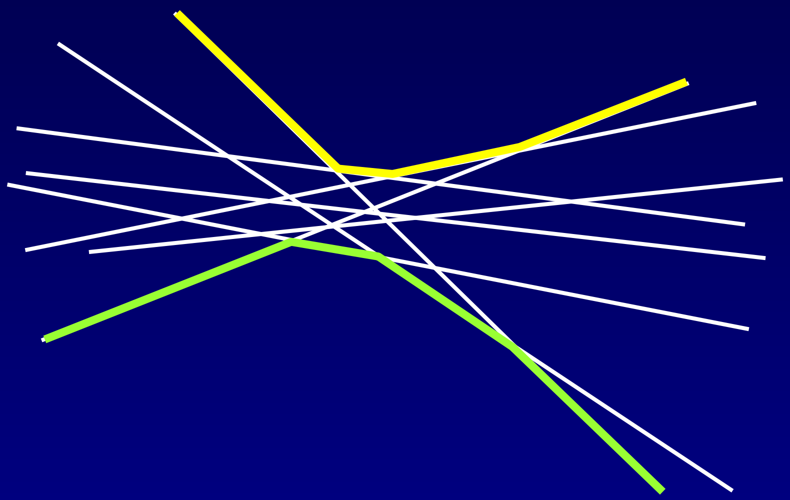


6. Line ℓ intersects segment $s \Leftrightarrow \ell^* \in s^*$.

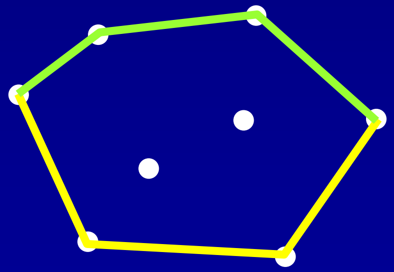
   Question: How can ℓ be located so that ℓ^* appears in the right side of the double wedge?

The Envelope Problem

- Problem: Find the (convex) lower/upper *envelope* of a set of lines ℓ_i – the boundary of the intersection of the halfplanes lying below/above all the lines.



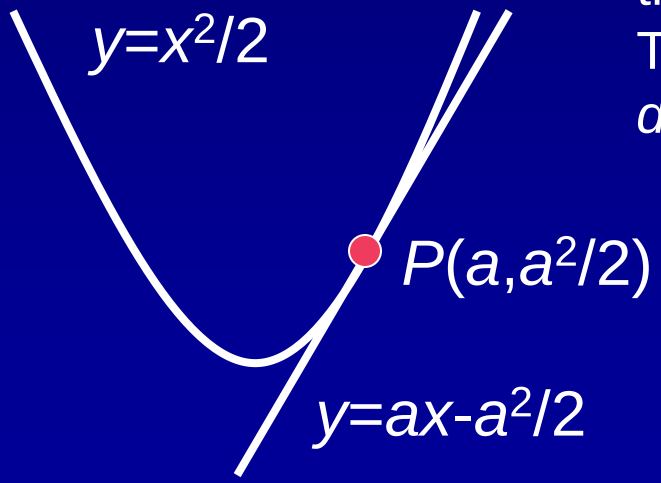
- **Theorem:** Computing the lower (upper) envelope is equivalent to computing the upper (lower) convex hull of the points ℓ_i^* in the dual plane.



- **Proof:** Using the order-preserving property.

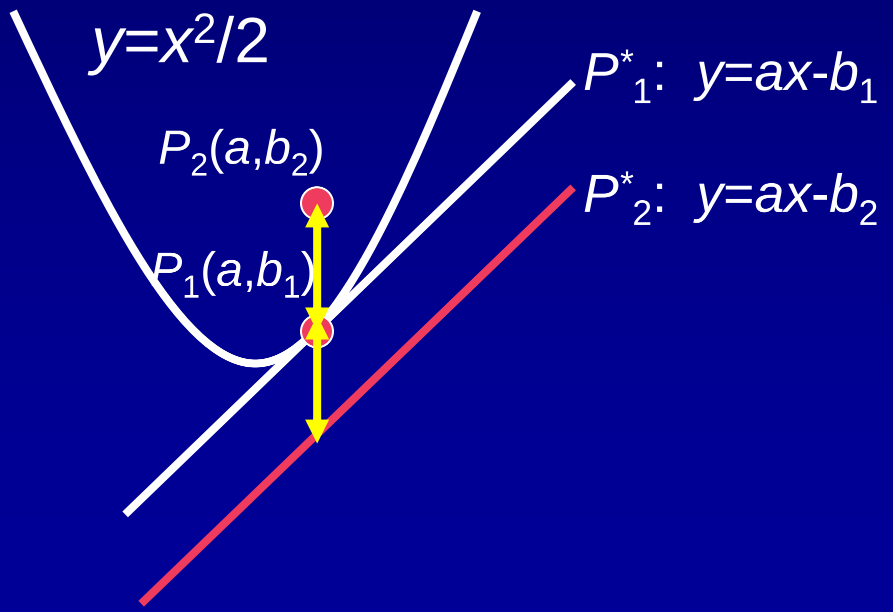
Parabola: Duality Interpretation

- Theorem: The dual line of a point on the parabola $y=x^2/2$ is the tangent to the parabola at that point.
- Proof:
  - Consider the parabola $y=x^2/2$. Its derivative is $y'=x$.
  - A point on the parabola: $P(a,a^2/2)$. Its dual: $y=ax-a^2/2$.
  - Compute the tangent at P: It is the line $y=cx+d$ passing through $(a,a^2/2)$ with slope $c=a$. Therefore, $a^2/2=a\cdot a+d$, that is, $d=-a^2/2$, so the line is $y=ax-a^2/2$.



Parabola: Duality Interpretation (cont.)

- And what about points not on the parabola?
- The dual lines of two points (a,b_1) and (a,b_2) have the same slope and the opposite vertical order with vertical distance $|b_1-b_2|$.



Yet Another Interpretation

Problem:

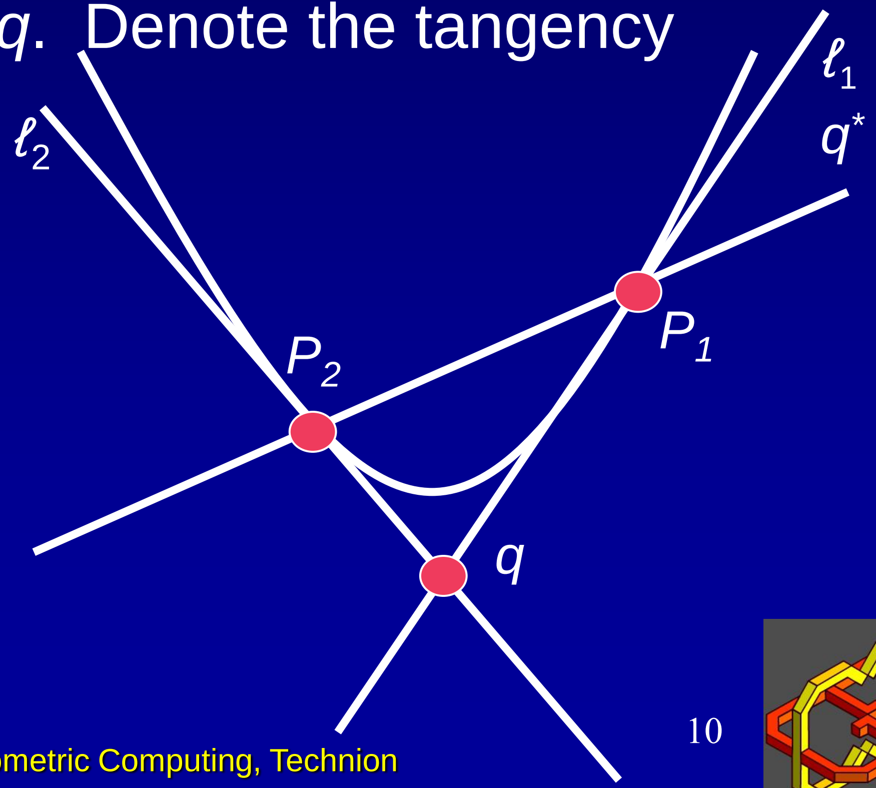
Given a point q below the parabola, what is the line q^*?

- Construct the two tangents ℓ_1, ℓ_2 to the parabola $y=x^2/2$ that pass through q. Denote the tangency points by P_1, P_2.
- Draw the line joining P_1 and P_2. This is q^*!
- Reason:

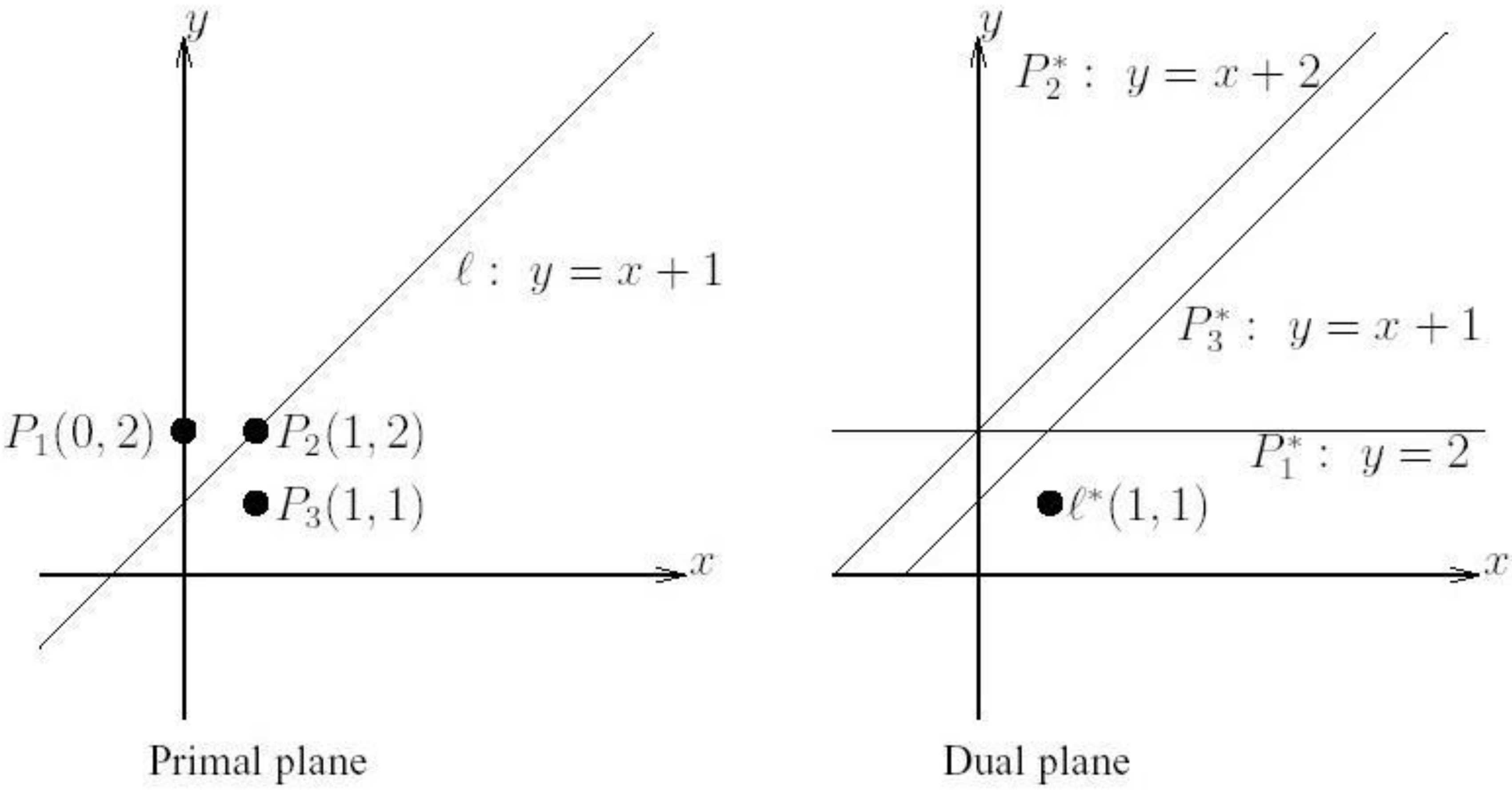
  q on $\ell_1 \rightarrow P_1 = \ell_1^*$ on q^*.

  q on $\ell_2 \rightarrow P_2 = \ell_2^*$ on q^*.

  Hence, $q^* = \overline{P_1P_2}$.



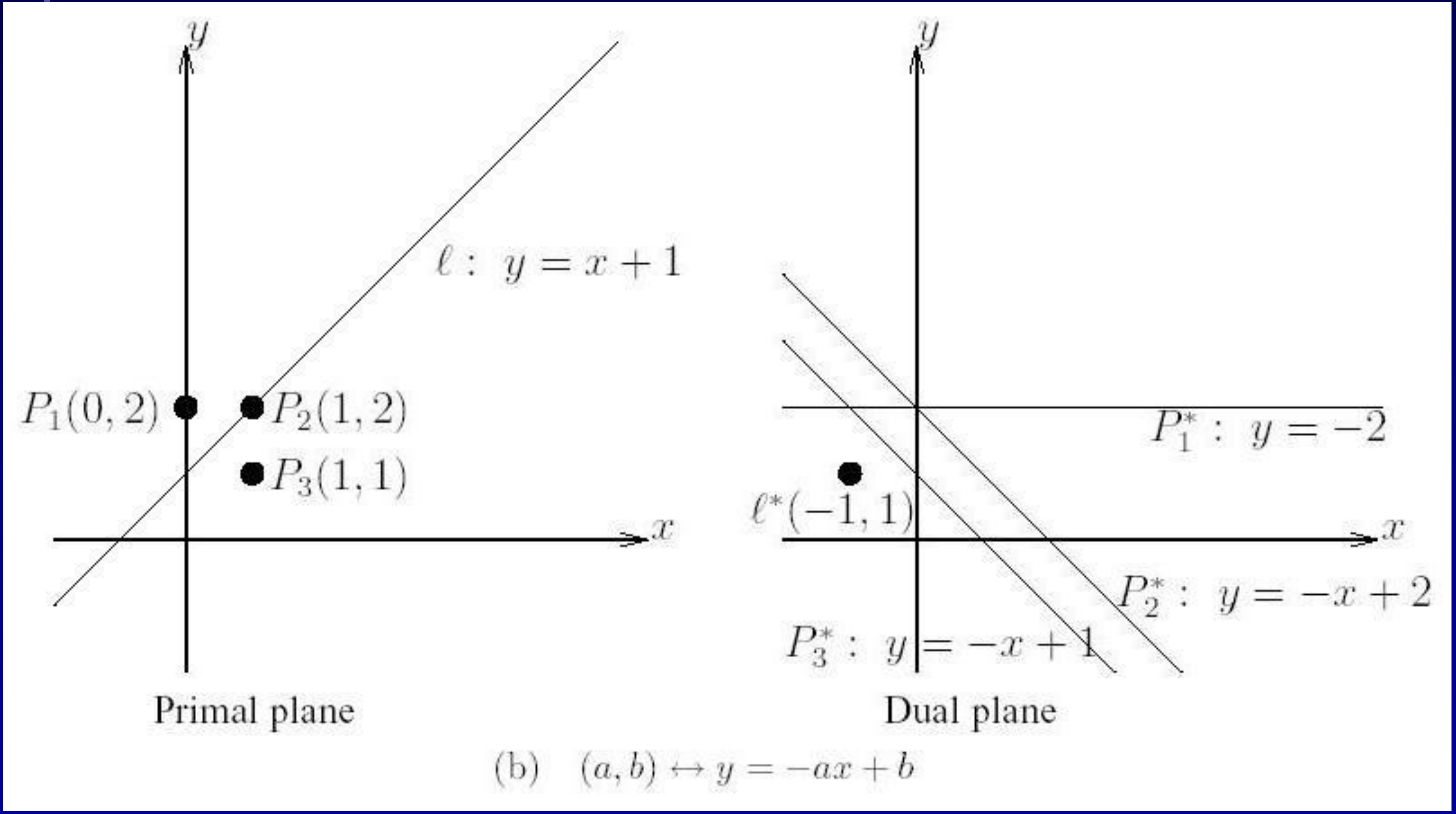
First Non-Preserving Duality



(a) $(a, b) \leftrightarrow y = ax + b$

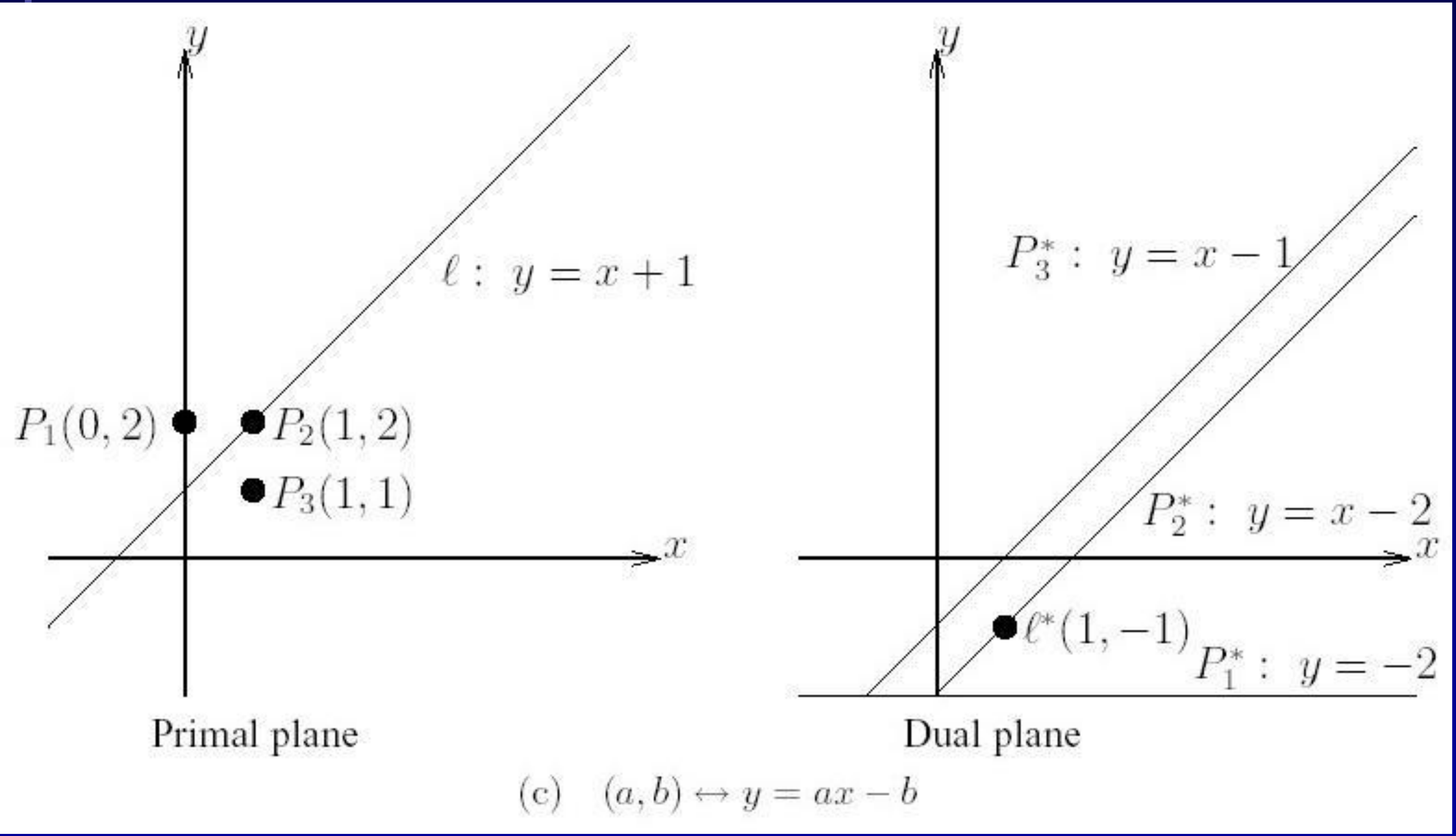
Note: Point ℓ^* is below **all** lines in the dual plane

Second Non-Preserving Duality



Note: Point ℓ^* is below **all** lines in the dual plane

The Preserving Duality



Point-line relations are preserved in the dual plane