Computational Geometry

Chapter 11

The Crossing-Number Lemma

On the Agenda

- The Crossing-Number Lemma
- Applications to combinatorial problems

Historical Perspective

Paul Erdős (born 1913 in Hungary, died 1996) was one of the greatest mathematicians of the 20th century. He published **thousands** of research papers during about 70 years, most of which dealing with problems in combinatorial geometry. Due to their difficulty, they got the nickname "Hard Erdős Problems." In 1982/3, the so-called **crossing-number lemma**, motivated by optimization problems in chip design, was proven. Only in 1998 Székely discovered that many hard Erdős problems can be solved (at least partially, but yielding no worse bounds) by ridiculously simple applications of this lemma. This opened a new era in combinatorial geometry, e.g., for proving a mile-stone upper bound on the complexity of the k^{th} level in an arrangement of n lines.

The Crossing Number

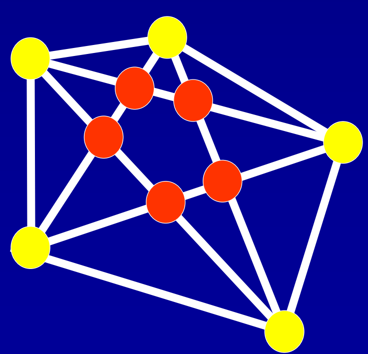
- The *crossing number* of a graph *G*, #cr(*G*), is the minimum number of edge crossings in a planar drawing of *G*.
- Corollary of Euler's formula: In every simple* planar graph $e \leq 3v-6$ (where *e* and *v* are the numbers of edges and vertices, respectively).
- Hence a graph in which $e > 3v-6$ cannot be planar. For example:

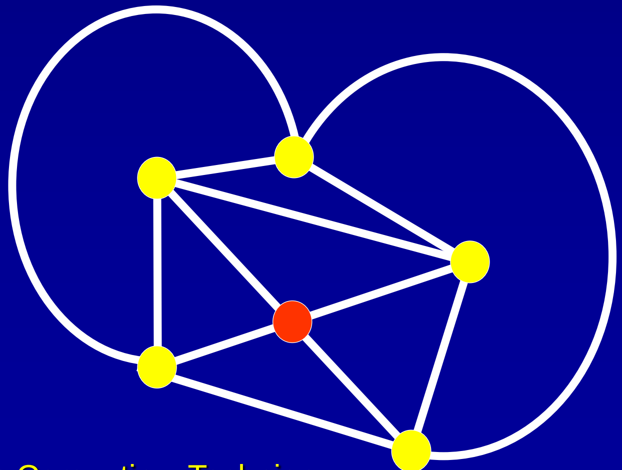
$v = 5$

$3v-6 = 9$

$e = 10$

#cr = 1



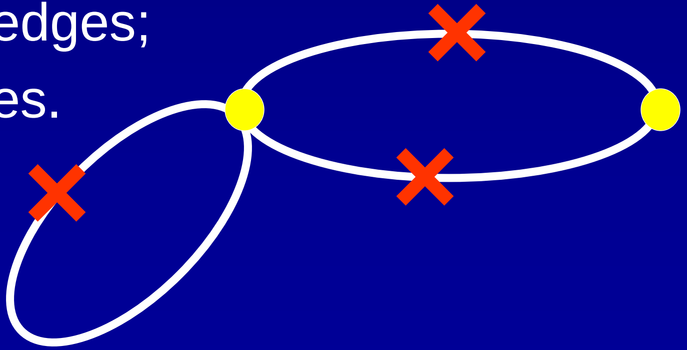


The Crossing-Number Lemma

- [Ajtai, Chvátal, Newborn, and Szemerédi, 1982] and [Leighton, 1983].
  Originally proven by induction on the graph complexity.
- Let G be a **simple** graph with v vertices and $e \geq 4v$ edges. Then:

$$\#\mathrm{cr}(G) = \Omega(e^3 / v^2)$$

- Remark: "Simple" means
  - No parallel edges;
  - No self edges.



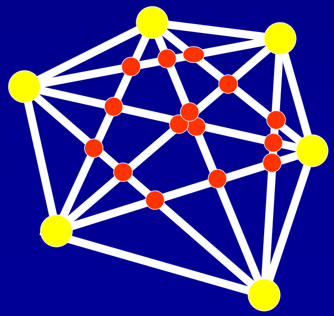
A Probabilistic Proof "From the Book" (Chazelle, Sharir, Welzl)

- Consider a planar embedding of a graph with v vertices, e edges, and c = #cr pairs of crossing edges.
- By Euler's formula $c \geq e-(3v-6) > e-3v$. (Why?)
- Choose a random subset of the vertices, each vertex with probability p (to be defined later).
- The expected number of vertices, edges, and crossings in the induced subgraph are pv, p^2e, and p^4c, respectively.
- That is, $p^4c > p^2e - 3pv$ (why?). Hence, $c > e/p^2 - 3v/p^3$. Choosing $p = 4v/e$ (thus, $0 \leq p \leq 1$ as needed) yields $c > e^3 / (16v^2) - 3e^3 / (64v^2) = e^3 / (64v^2)$.
- Question: Why is this a proof?
- The constant 1/64 can be improved (enlarged) from $1/64 \approx 0.0156$ to $4/135 \approx 0.0296$ (and even more).

That is, why does this **probabilistic** proof show that the claim ***always*** holds?

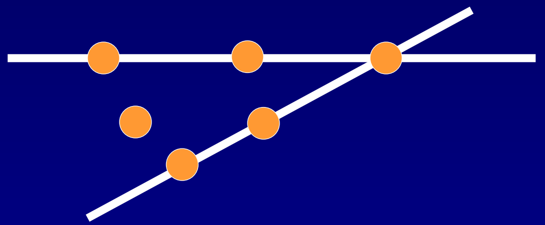
Application I: Segment Intersections

- Given a complete graph *G* with *n* points in the plane in general position (no three collinear points).
- **Problem:** What is the crossing number of *G*?
- **Simple upper bound:** $O(n^4)$ intersections. (Why?)
- **Lower bound** (by the lemma): $\Omega\left(\left(n^2\right)^3 / n^2\right) = \Omega\left(n^4\right)$
- That is, we have a tight bound of $\Theta(n^4)$ on the c.n. of G.
- Question: Why can we apply the lemma?
- Question: Does it matter if the graph is *geometric*? (A geometric graph is made of straight line-segments only.)



Application II: Point-Line Incidences

- Let P be a set of n distinct points and L a set of ℓ distinct lines.
- An *incidence* of P and L is a pair (p,q), where $p \in P$, $q \in L$, and p lies on q. #i(P,L) is the number of such incidences.

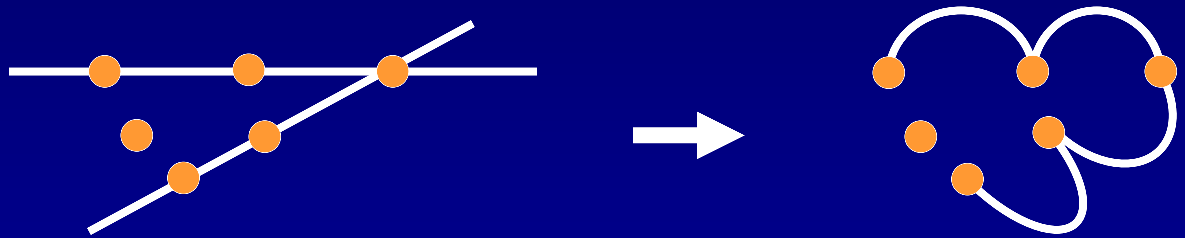


$n = 6$
$\ell = 2$
#i = 6

- The minimum possible value of #i(P,L) is obviously 0.
- What is the **maximum** possible value of #i(P,L)?
- Clearly, #i = $O(n\ell)$. Can we do better?
- **Theorem:** $$\#\mathrm{i} = O((n\ell)^{2/3} + n + \ell)$$
  (note the role of the $(n+\ell)$ term)

Proof of the P/L-I Theorem

- For a given point-set *P* and line-set *L*, construct a graph in which each point in *P* is a vertex, and an edge connects every pair of consecutive points along a line of *L*.



- For each line q, $e(q) = v(q)–1$. (Why?)
- Sum up over all lines in *L* to obtain $e = \#i–\ell$. (Why?)
- Trivially, in the graph $\#cr \le \ell^2$. (Why?)

$e = \#i - \ell$

Proof of the P/L-I Theorem (cont.)

Case 1: $e \leq 4n$

$\rightarrow 4n \geq \#i - \ell$

$\rightarrow \#i \leq 4n + \ell$

$\rightarrow \#i = O(n + \ell)$

Case 2: $e \geq 4n$

$\#cr = \Omega(e^3/n^2) = \Omega((\#i - \ell)^3/n^2)$

$\#cr = O(\ell^2)$

$\rightarrow (\#i - \ell)^3 = O(n^2\ell^2)$

$\rightarrow \#i = O((n\ell)^{2/3} + \ell)$

$$\#i = O((n\ell)^{2/3} + n + \ell)$$

Note: in the special case $\ell = n$, $\#i = O(n^{4/3})$.

Application III (Number Theory)

- Let A be a set of n **distinct** integer numbers.
- $A \cdot A + A$ is the set of integers created by multiplying two elements from A, and adding another element.
- Clearly,

  $k = |A \cdot A + A| = \Omega(n)$ (but not completely trivially, since, e.g., $(-2) \cdot (-2) + (-2) = 1 \cdot 1 + 1$, so why?),

  and

  $k = O(n^3)$. (Why?)

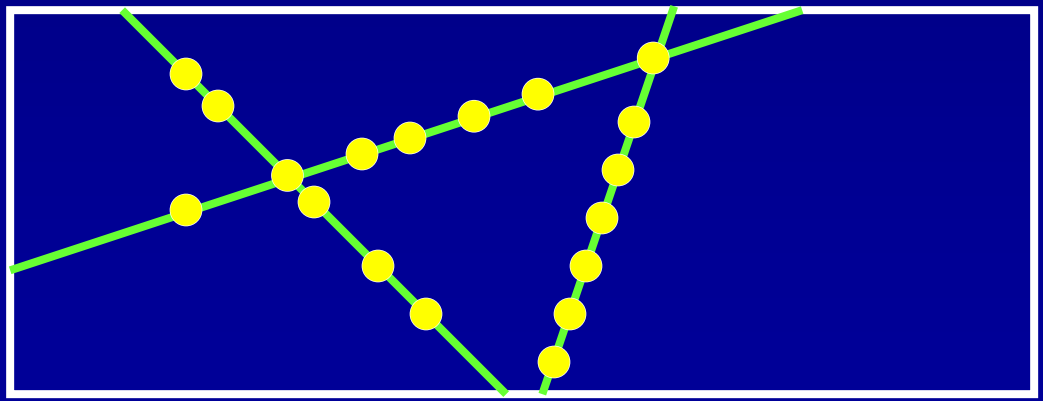
- How small can k really be?

Solution

- Let S be a set of points: $S = \{(x,y) \mid x \in A,\ y \in A \cdot A + A\}$. Obviously, $|S| = nk$.
- Draw all the lines of the form $y = a_i x + a_j$, where $a_i, a_j \in A$.
- Observations (justify!):

  1. There are exactly n^2 such lines;

  2. Each such line passes through exactly n points of S.
- Therefore, $\#i = n^3$.



Applying the Crossing-Number Lemma

- Recall: nk points, n^2 lines.
- According to the point/line-incidences theorem,
  $n^3 = \#i = O(((nk)n^2)^{2/3} + n^2 + nk) = O(n^2k^{2/3} + n^2 + nk)$.
- But: $n^2 = O(n^2k^{2/3})$ and
  $k \le n^3 \xrightarrow{x^{1/3}} k^{1/3} \le n \xrightarrow{\cdot nk^{2/3}} nk \le n^2k^{2/3}$!
- That is,
  $n^3 = O(n^2k^{2/3}) \rightarrow k^{2/3} = \Omega(n) \rightarrow k = \Omega(n^{3/2})$.

So these two terms are redundant!