2-Point Site Voronoi Diagrams

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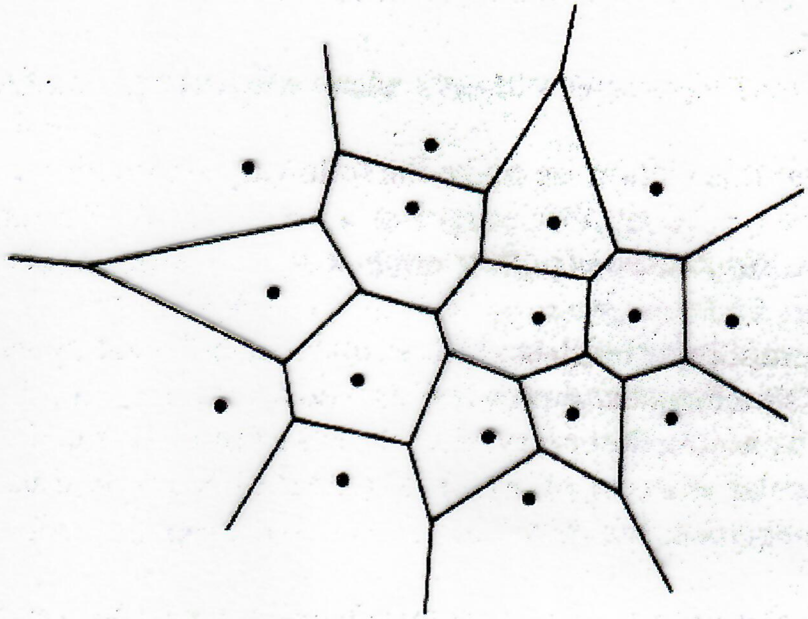
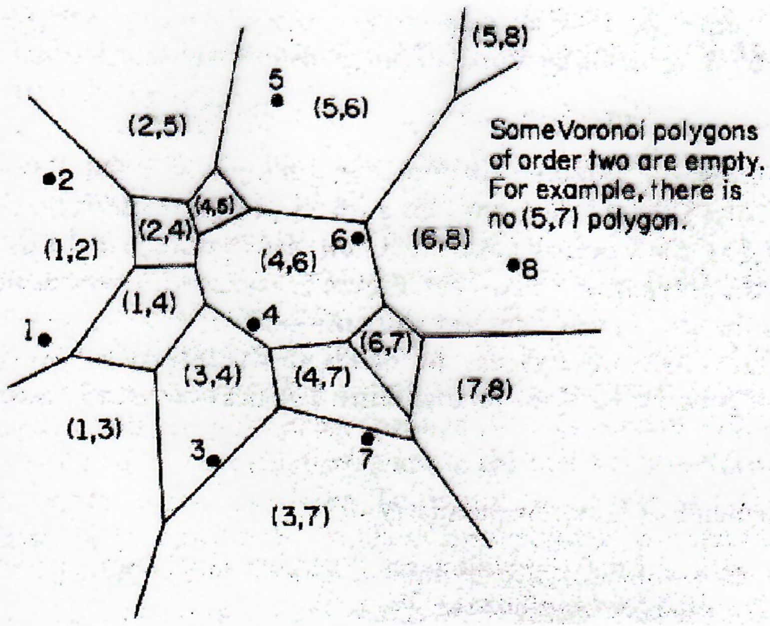
Ordinary Voronoi Diagrams of Points in $\mathbb{R}^2$

1st-order diagram



2nd-order diagram



(from [Preparata-Shamos 85, pp. 199, 235])

2-Site Distance Functions

1. **Sum of Distances**:
   $\mathcal{S}(v,(p,q)) = d(v,p) + d(v,q)$;
   Product of Distances:
   $\mathcal{M}(v,(p,q)) = d(v,p) \cdot d(v,q)$.
2. **Triangle Area**: $\mathcal{A}(v,(p,q)) = A(v,p,q)$.
3. **Distance from a Line**:
   $\mathcal{L}(v,(p,q)) = \min_{u \in \ell_{pq}} d(v,u)$;
   Distance from a Segment:
   $\mathcal{G}(v,(p,q)) = \min_{u \in \overline{pq}} d(v,u)$.
4. **Difference between Distances**:
   $\mathcal{D}(v,(p,q)) = |d(v,p) - d(v,q)|$.
5. **Triangle Perimeter**:
   $\mathcal{P}(v,(p,q)) = d(p,q) + d(v,p) + d(v,q)$.
6. **Circumcircle Radius**: $\mathcal{R}(v,(p,q)) = \frac{d(v,p)d(v,q)d(p,q)}{4\sqrt{s(s-d(v,p))(s-d(v,q))(s-d(p,q))}}$,
   where $s = \mathcal{P}(v,(p,q))/2$.

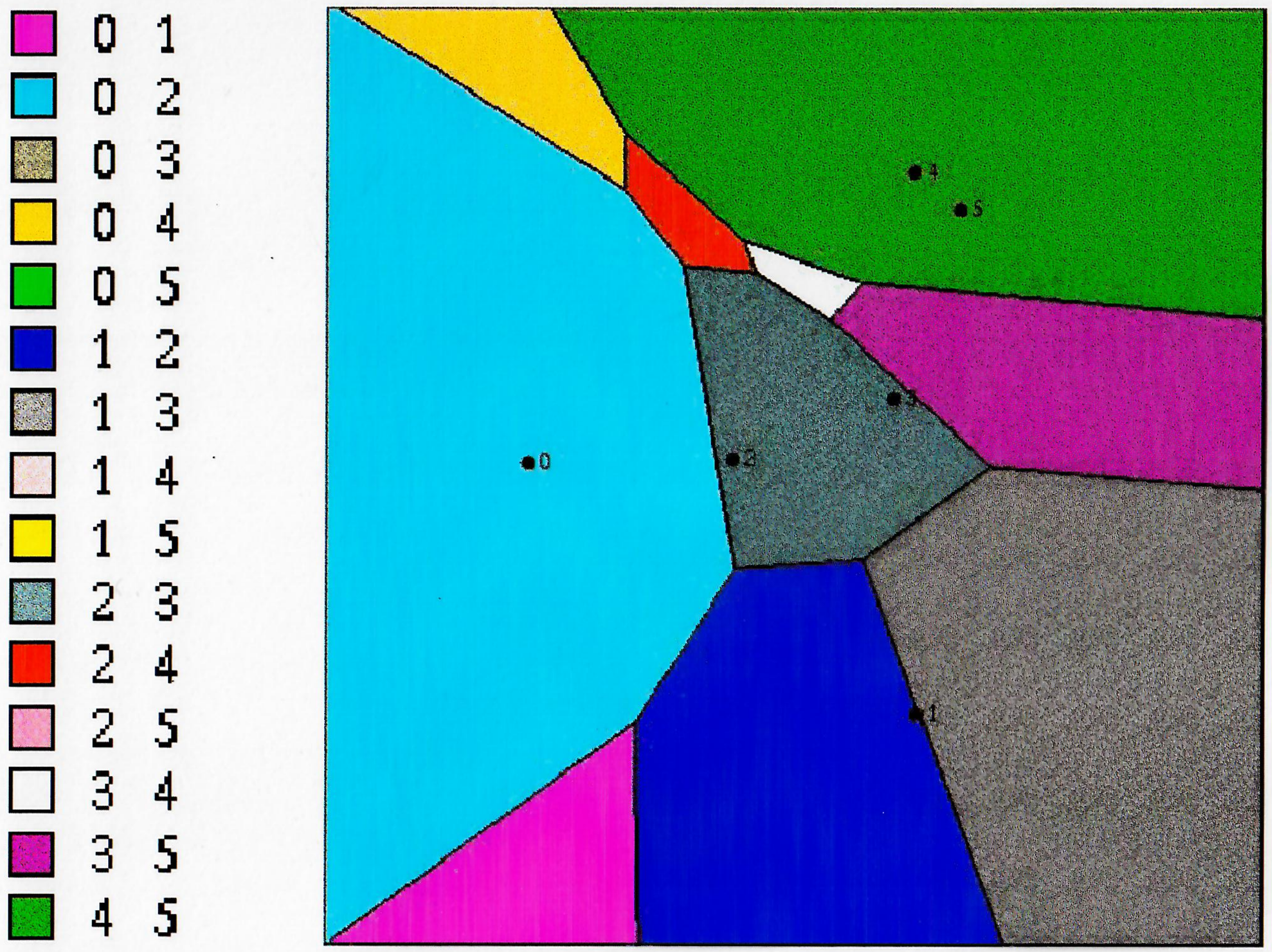
Results (worst-case diagram complexities)

$\mathcal{F}$	$\lvert V_{\mathcal{F}}^{(N)}(S)\rvert$ (NN Diagram)	$\lvert V_{\mathcal{F}}^{(F)}(S)\rvert$ (FN Diagram)
$\mathcal{S}$, $\mathcal{M}$	$\Theta(n)^*$	$\Theta(n)$
$\mathcal{A}$	$\Theta(n^4)^*$	$\Theta(n^2)$
$\mathcal{L}$	$\Theta(n^4)^*$	$\Theta(n^2)$
$\mathcal{G}$	$\Theta(n^4)^*$	$\Theta(n)$
$\mathcal{D}$	$\Omega(n^4)^*$, $O(n^{4+\varepsilon})$	$\Theta(n^2)$
$\mathcal{P}$	?	?
$\mathcal{R}$	?	?

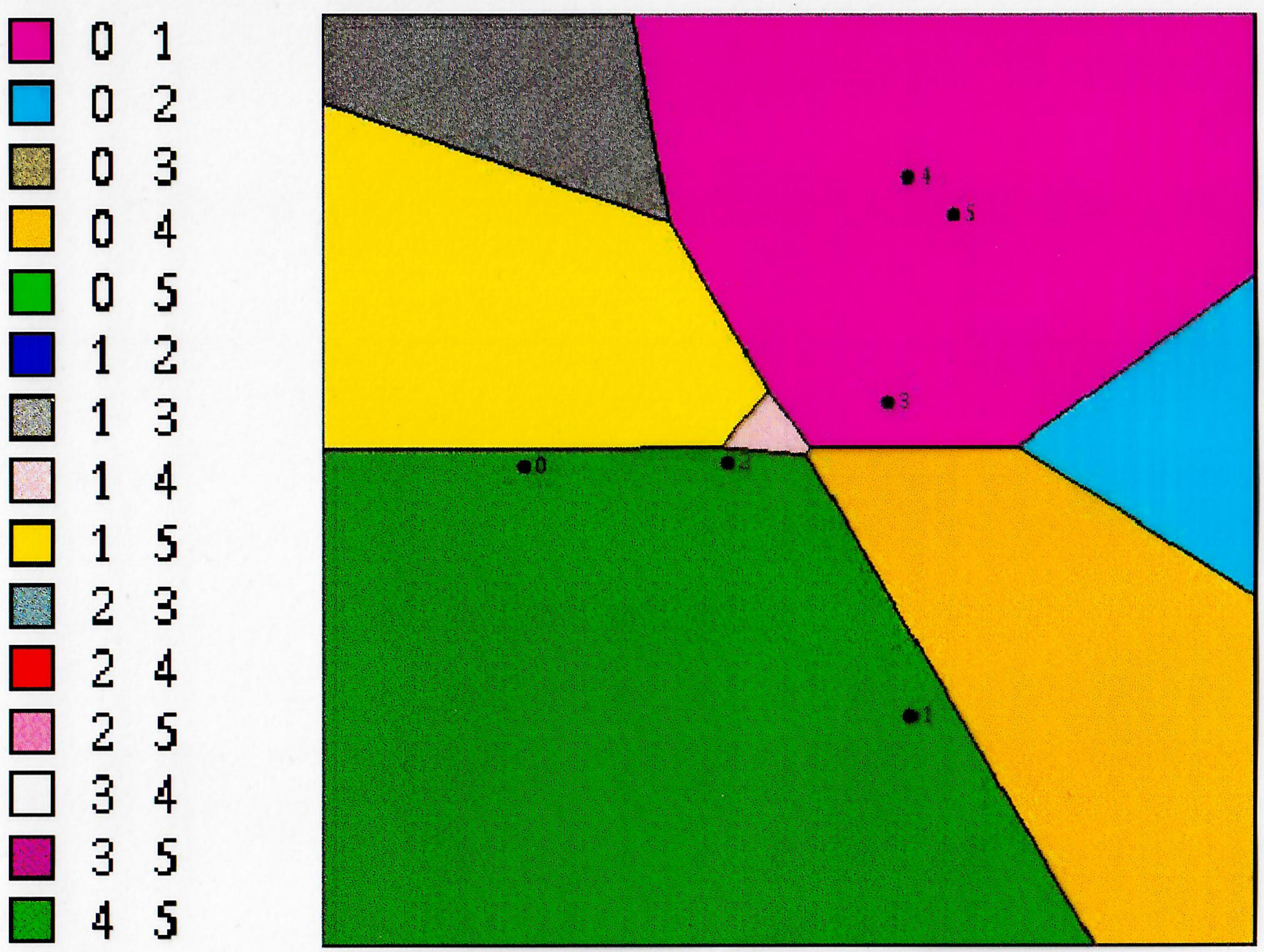
*Always

Legend:
S: a set of n points in $\mathbb{R}^2$;
$V_{\mathcal{F}}^{(N|F)}(S)$: Nearest/furthest-neighbor Voronoi diagram w.r.t. $\mathcal{F}$.

Point set



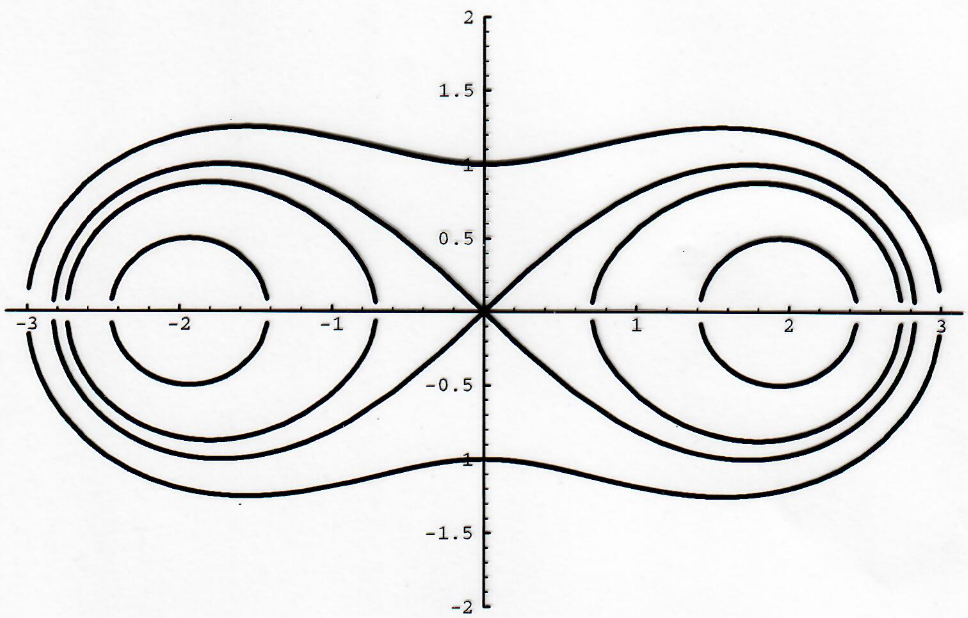
Sum (or product) of distances: NN Voronoi



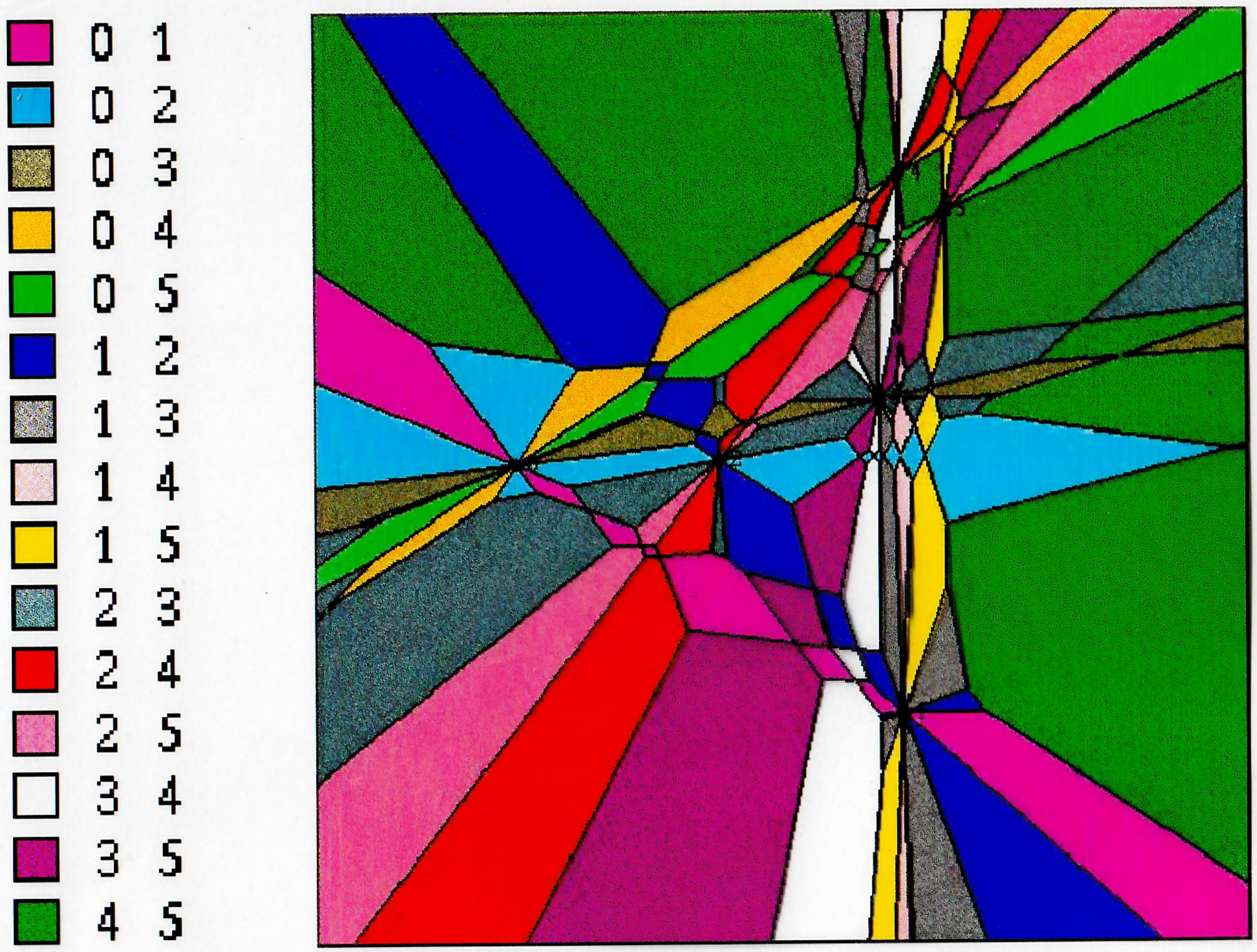
Sum (or product) of distances: FN Voronoi

Sum and Product of Distances

Growing shape:
Sum: an ellipse;
Product: a pair of leaves.



Theorem:
$V^{(N|F)}_{\mathcal{S}|\mathcal{M}}(S)$ are identical to the regular (one-site, Euclidean metric) 2nd-order $V^{(N|F)}(S)$.
(Therefore $|V^{(N|F)}_{\mathcal{S}|\mathcal{M}}(S)| = \Theta(n)$.)



Triangle area: NN Voronoi diag.

Triangle Area

Growing shape: an infinite strip; speed inversely-proportional to $d(p, q)$.

Theorem: $|V_{\mathcal{A}}^{(N)}(S)| = \Theta(n^4)$.

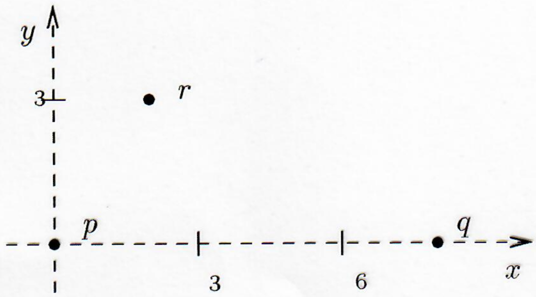
Proof:

Lower bound: $\Theta(n^4)$ intersection points of lines.

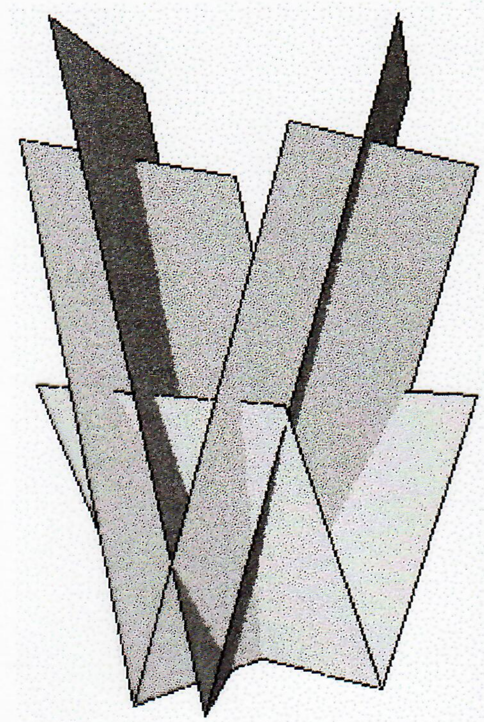
Upper bound: using the zone theorem [Edel.-Seidel-Sharir 93].

Triangle Area (cont.)

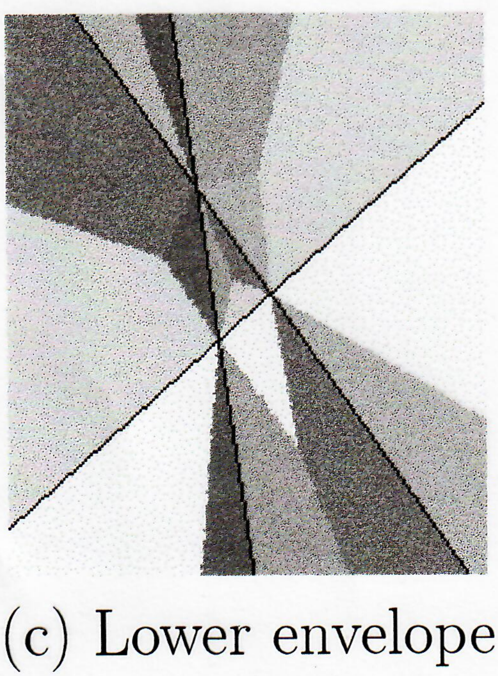
Computing $V_{\mathcal{A}}$:



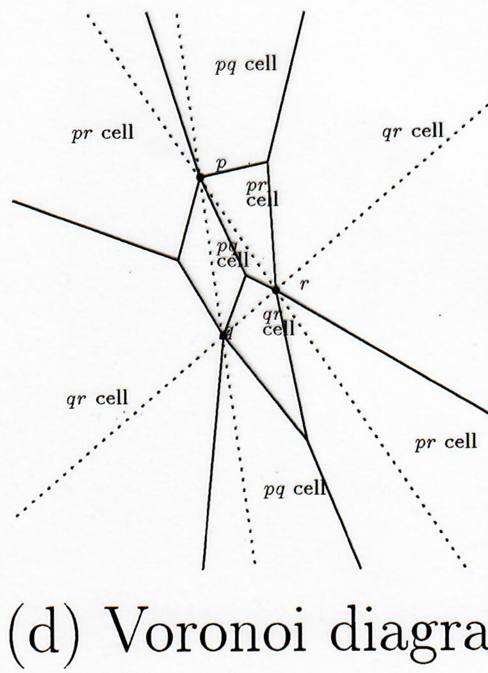
(a) Point set



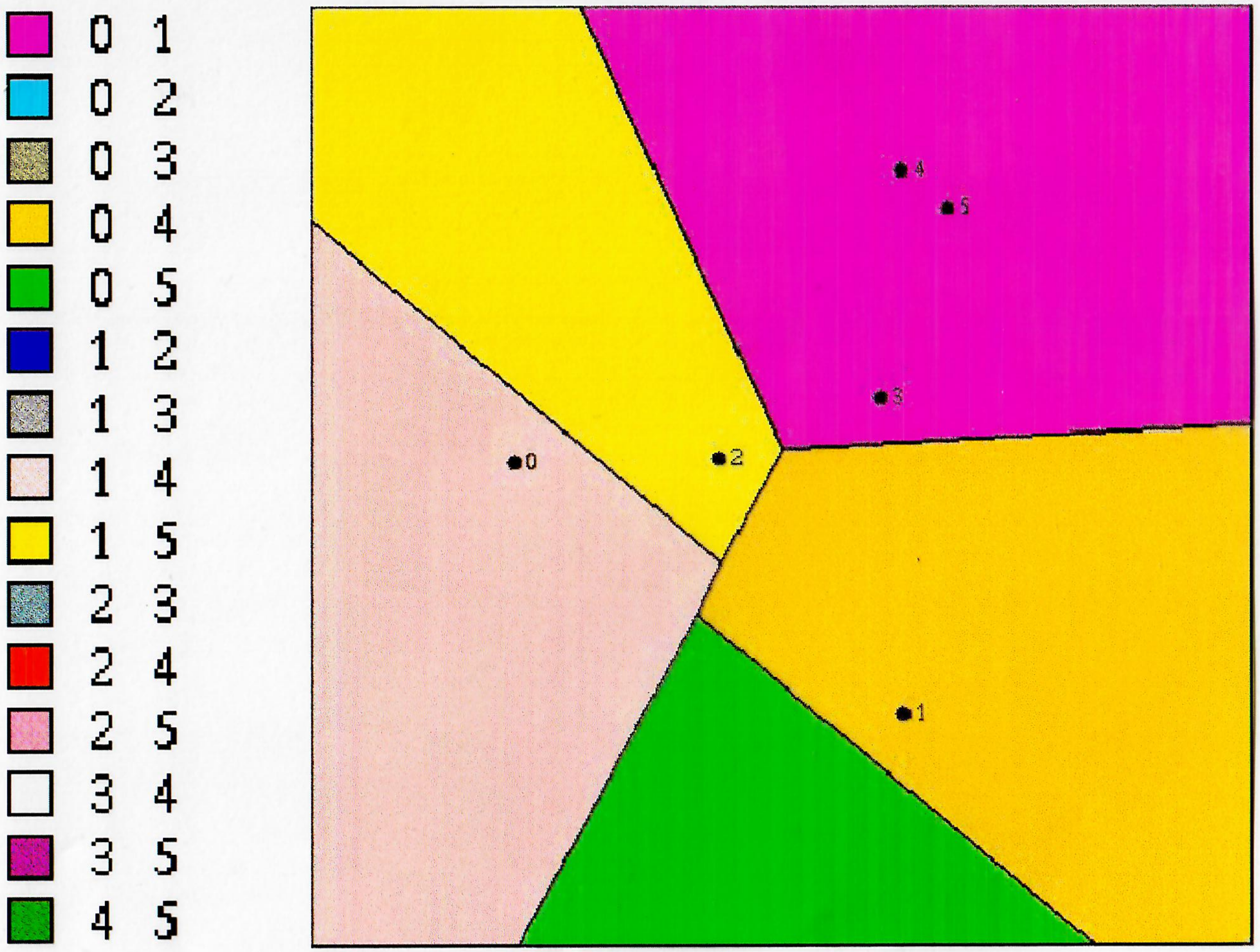
(b) Surfaces



(c) Lower envelope



(d) Voronoi diagram

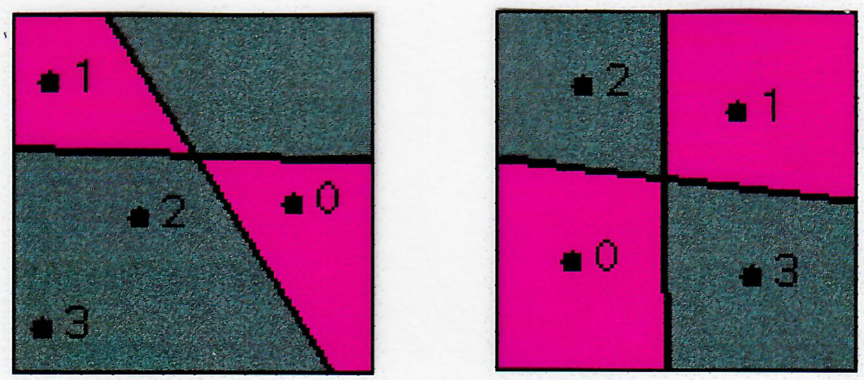


Triangle area: FN Voronoi diag.

Triangle Area (cont.)

Theorem: $|V_{\mathcal{A}}^{(F)}(S)| = \Theta(n^2)$.

Proof:
Lower bound: by example.
Upper bound:
each region has ≤ 2 convex cells; also: upper envelope of an arrangement of $\Theta(n^2)$ planes in $\mathbb{R}^3$ (using [Sharir-Agarwal 95]).



(0,1)–(2,3) bisector

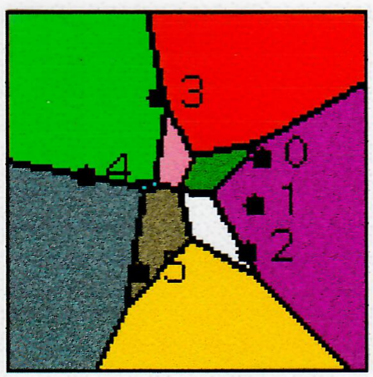
Theorem:
$V_{\mathcal{A}}^{(N|F)}(S)$ can be computed in $O(n^{4|2} \log n)$ time and $O(n^{4|2})$ space. (By a divide-and-conquer lower-envelope algorithm.)

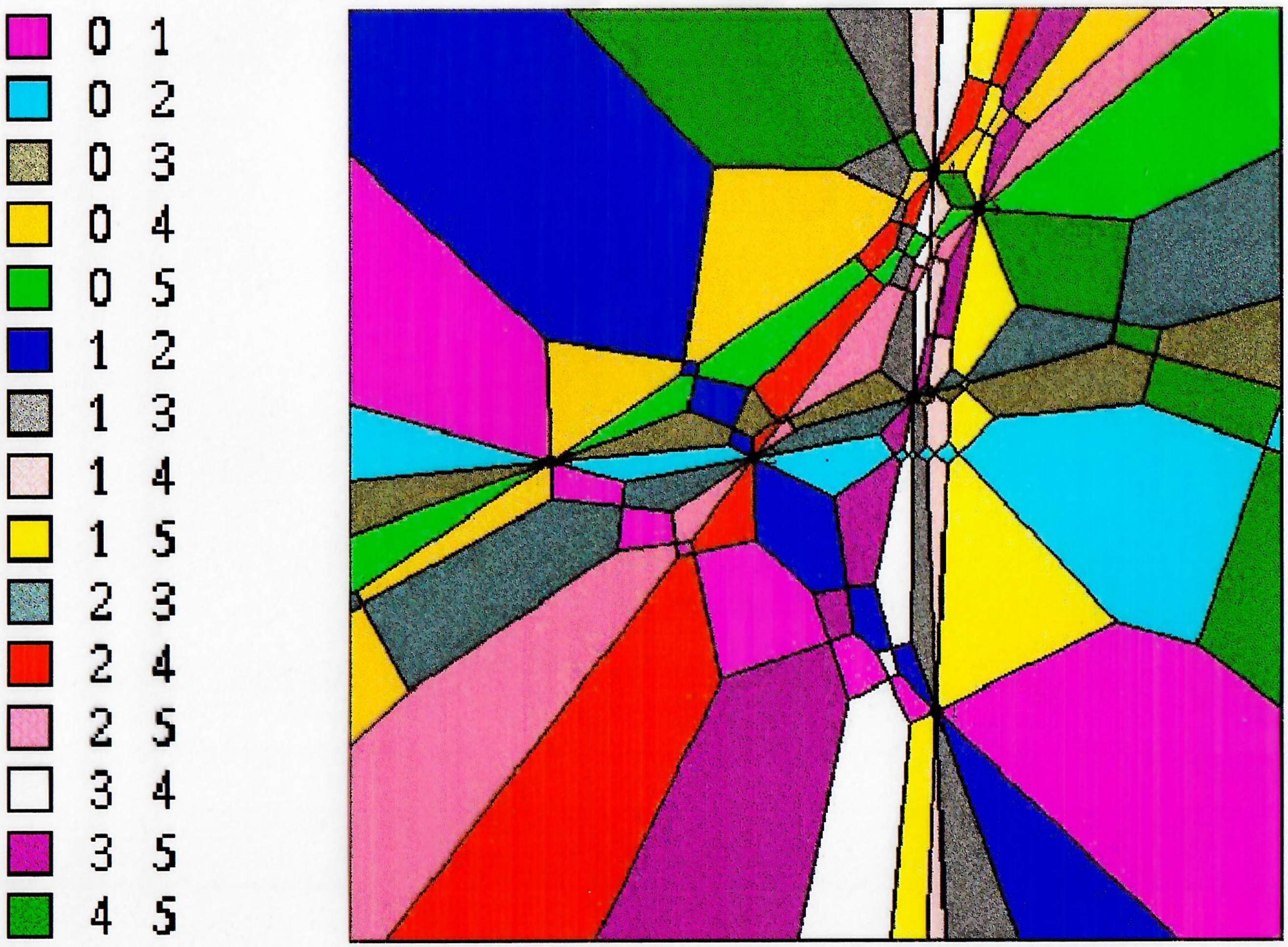
Triangle Area (cont.)

Theorem:

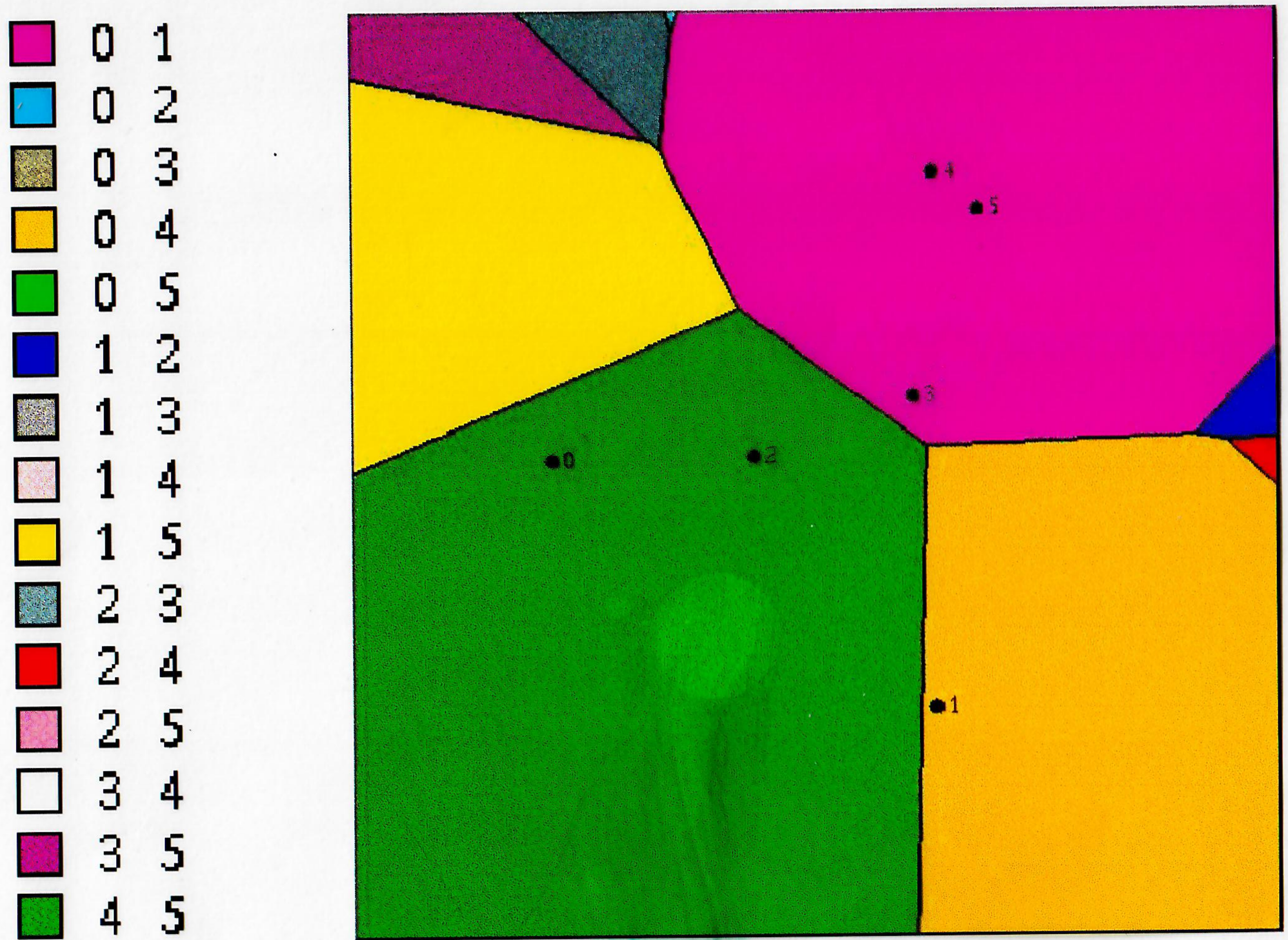
In $V_{\mathcal{A}}^{(F)}(S)$:

1. Only hull points (not on a hull edge) have non-empty regions.
2. Only antipodal points have infinite cells.





Distance from a line: NN Voronoi diag.



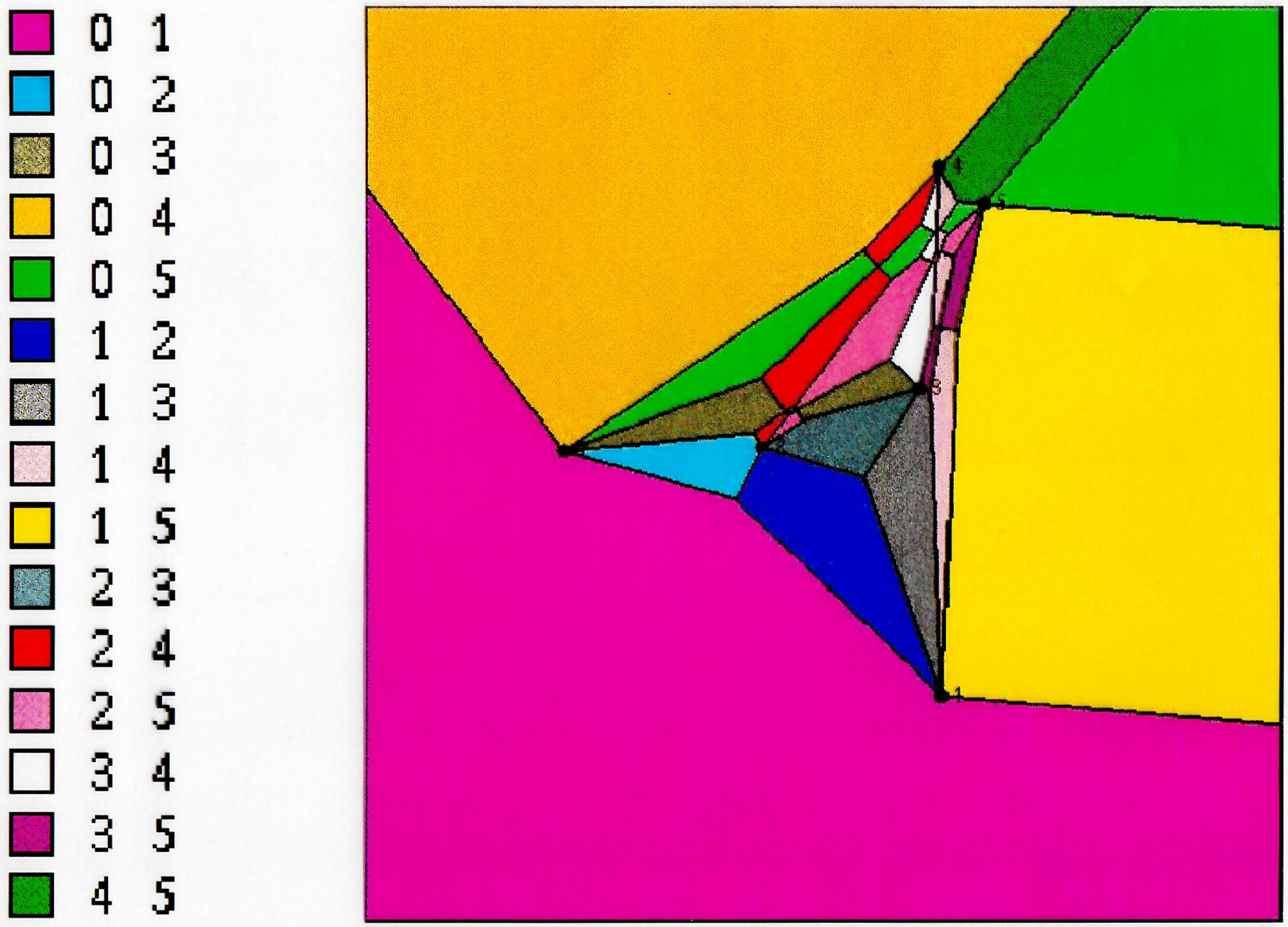
Distance from a line: FN Voronoi diag.

Distance from a Line

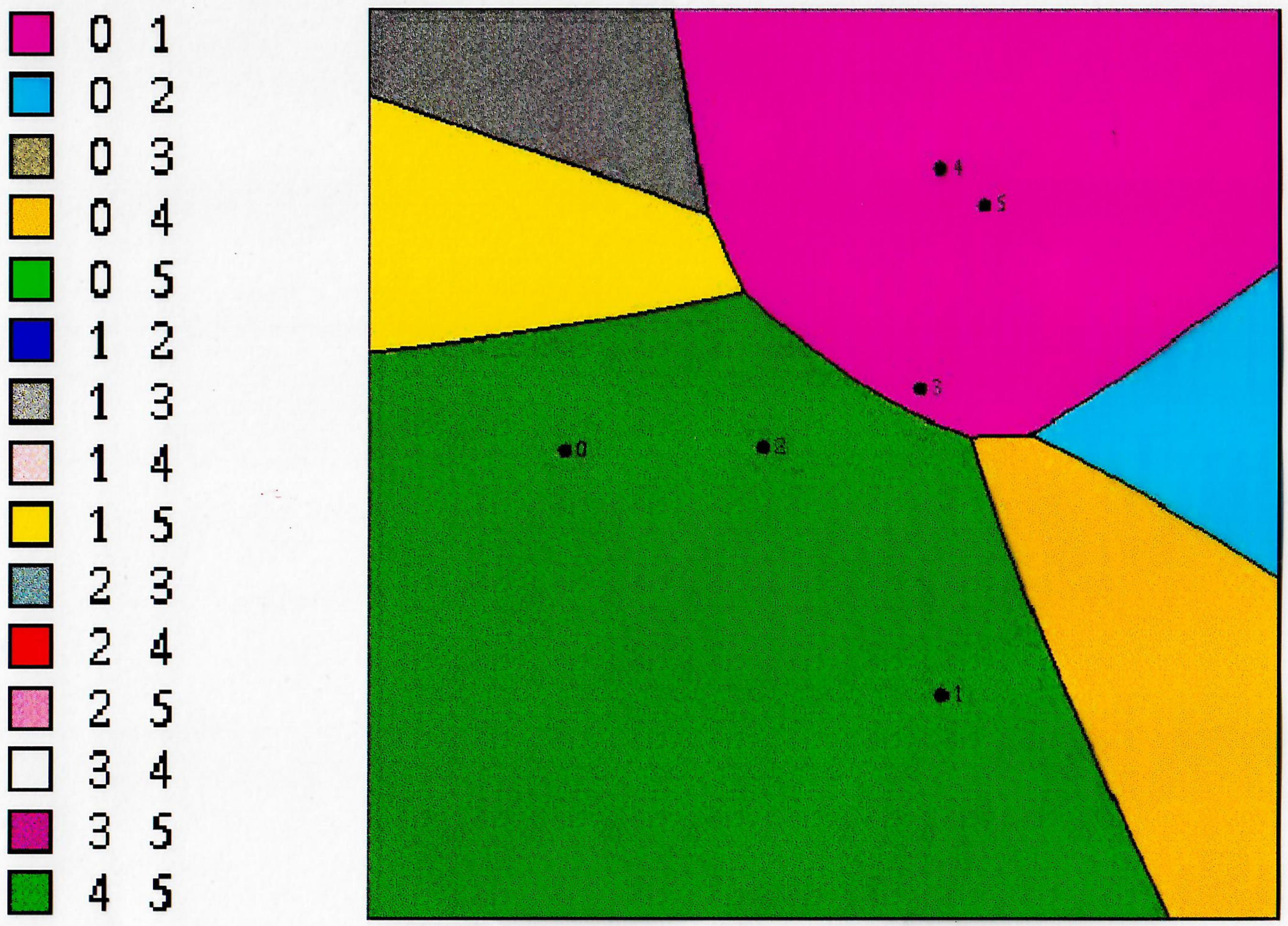
Growing shape: a strip.

Theorem:
$|V_{\mathcal{L}}^{(N)}(S)| = \Theta(n^4)$ and $|V_{\mathcal{L}}^{(F)}(S)| = \Theta(n^2)$.

Proof:
Same as for the triangle-area function.
(All the half-planes now have slope $\pi/4$.)



Distance from a segment: NN Voronoi diag.



Distance from a segment: FN Voronoi diag.

Distance from a Segment

Growing shape: a hippodrome.

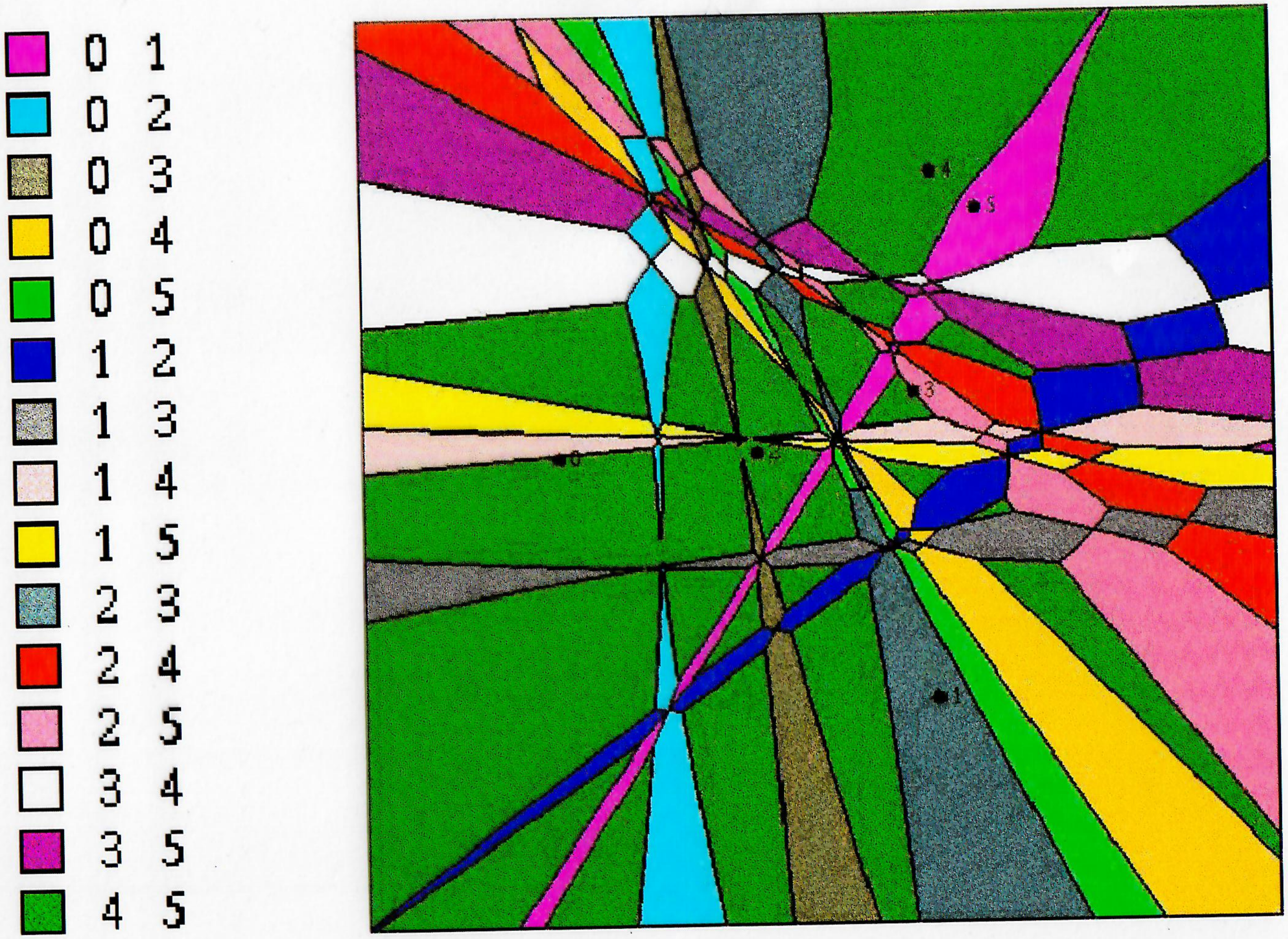
Theorem: $|V_{\mathcal{G}}^{(N)}(S)| = \Theta(n^4)$.

Proof:
Lower bound: $\Theta(n^4)$ segment-intersection points (using [Leighton 83]).
Upper bound: by splitting segments (using [Lee-Drysdale 81]).

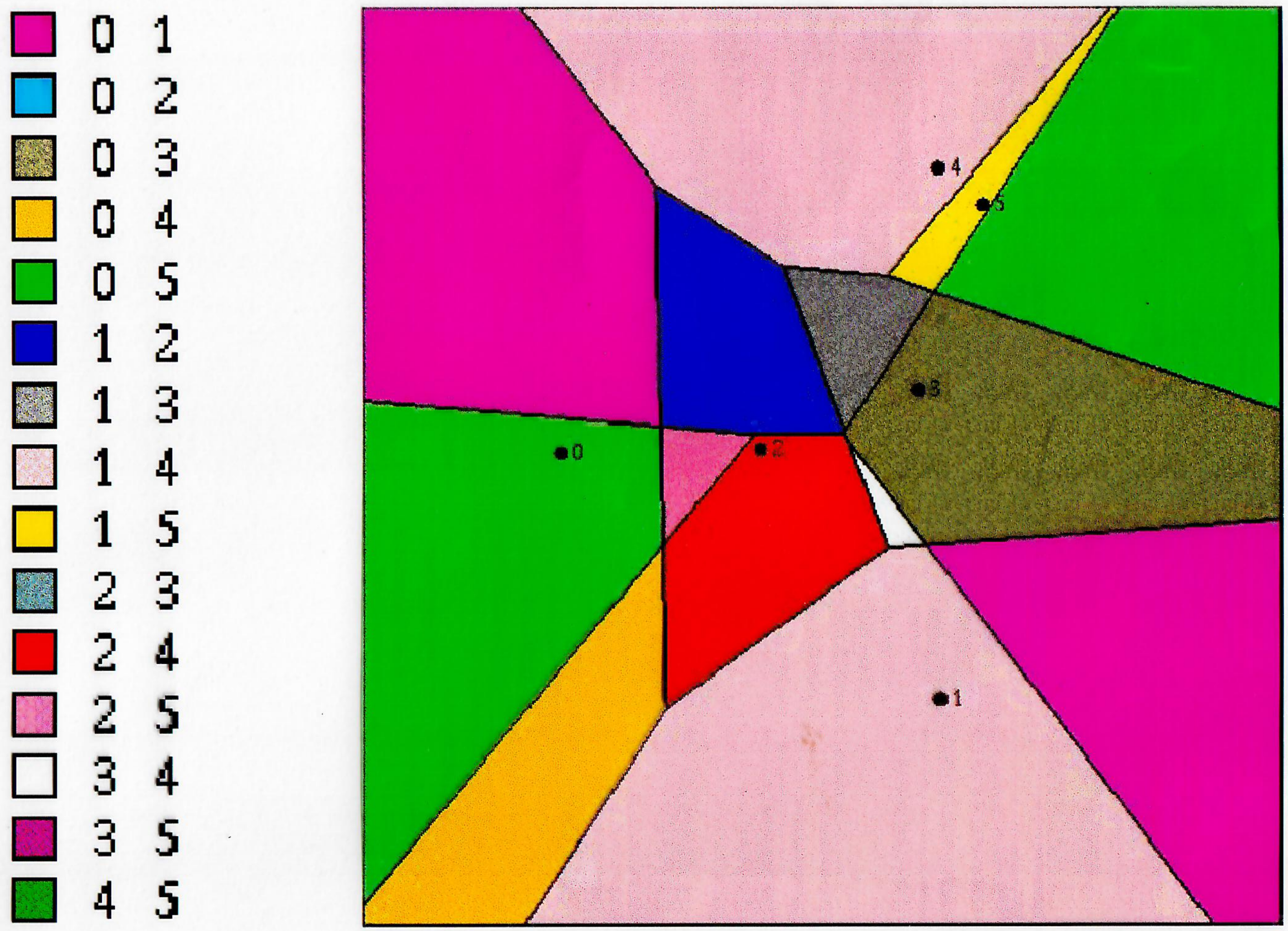
Theorem: $|V_{\mathcal{G}}^{(F)}(S)| = \Theta(n)$.

Proof: By "merging" the outer and inner convex hulls: $\mathrm{CH}(S)$ and $\mathrm{CH}(S \setminus \mathrm{CH}(S))$.

Theorem: $V_{\mathcal{G}}^{(N)}(S)$ can be computed in $O(n^4 \log n)$ time and $O(n^4)$ space. (By either the D&C envelope algorithm, or by algorithms of [Fortune 86] and [Yap 87].)



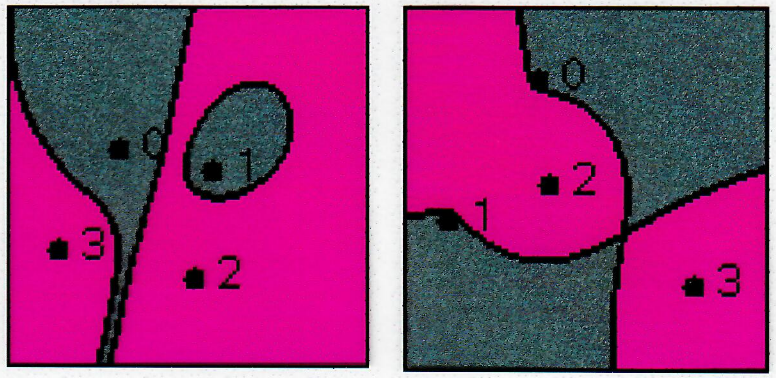
Difference of distances: NN Voronoi diag.



Difference of distances: FN Voronoi diag.

Difference between Distances

Growing shape: a "strip" bounded by two hyperbolas.



(0,1)–(2,3) bisector

Theorem: $V_{\mathcal{D}}^{(N)}(S) = \Omega(n^4), O(n^{4+\varepsilon})$.

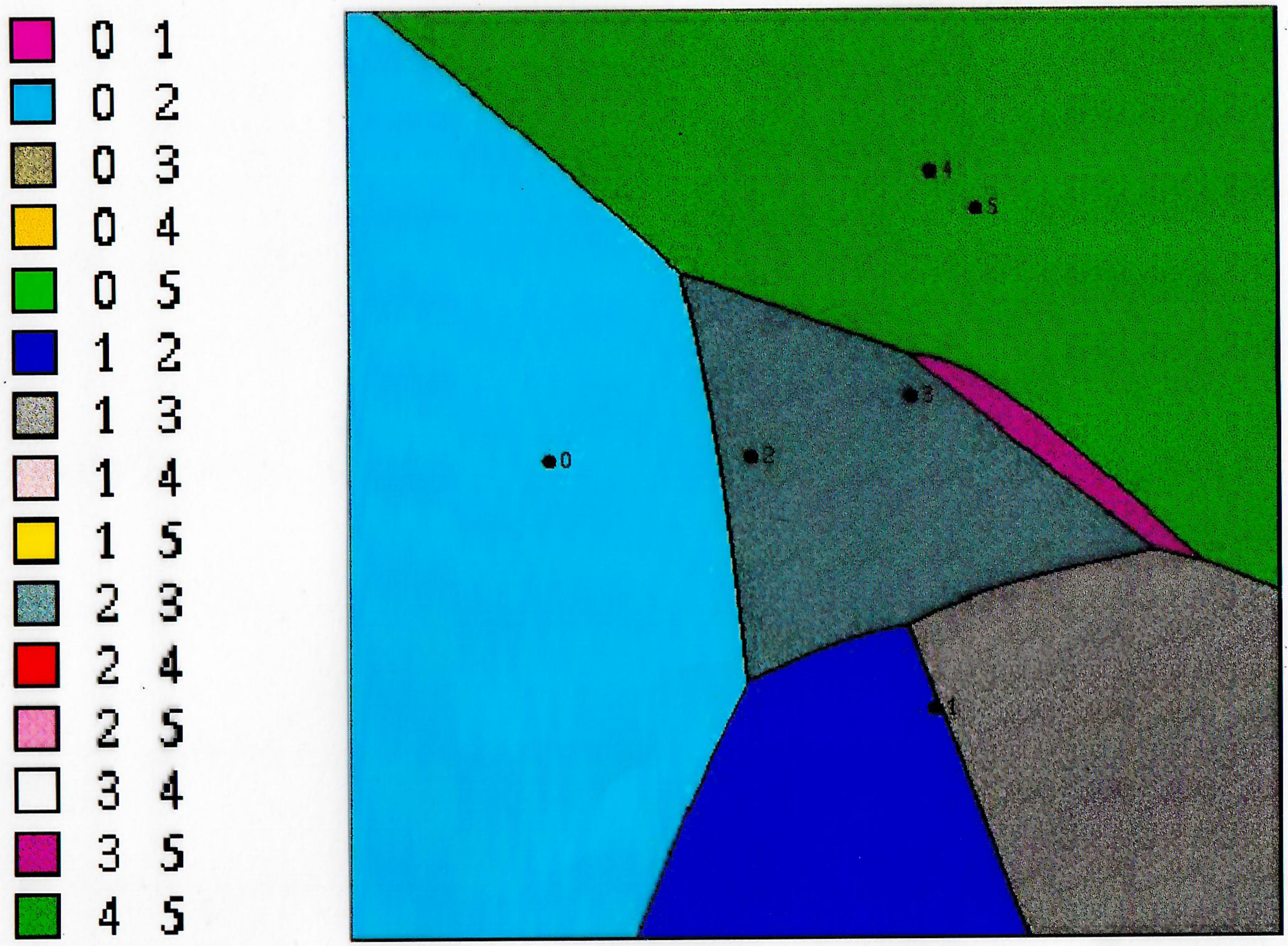
Proof:

Lower bound: $\Theta(n^4)$ intersection points between segment bisectors.

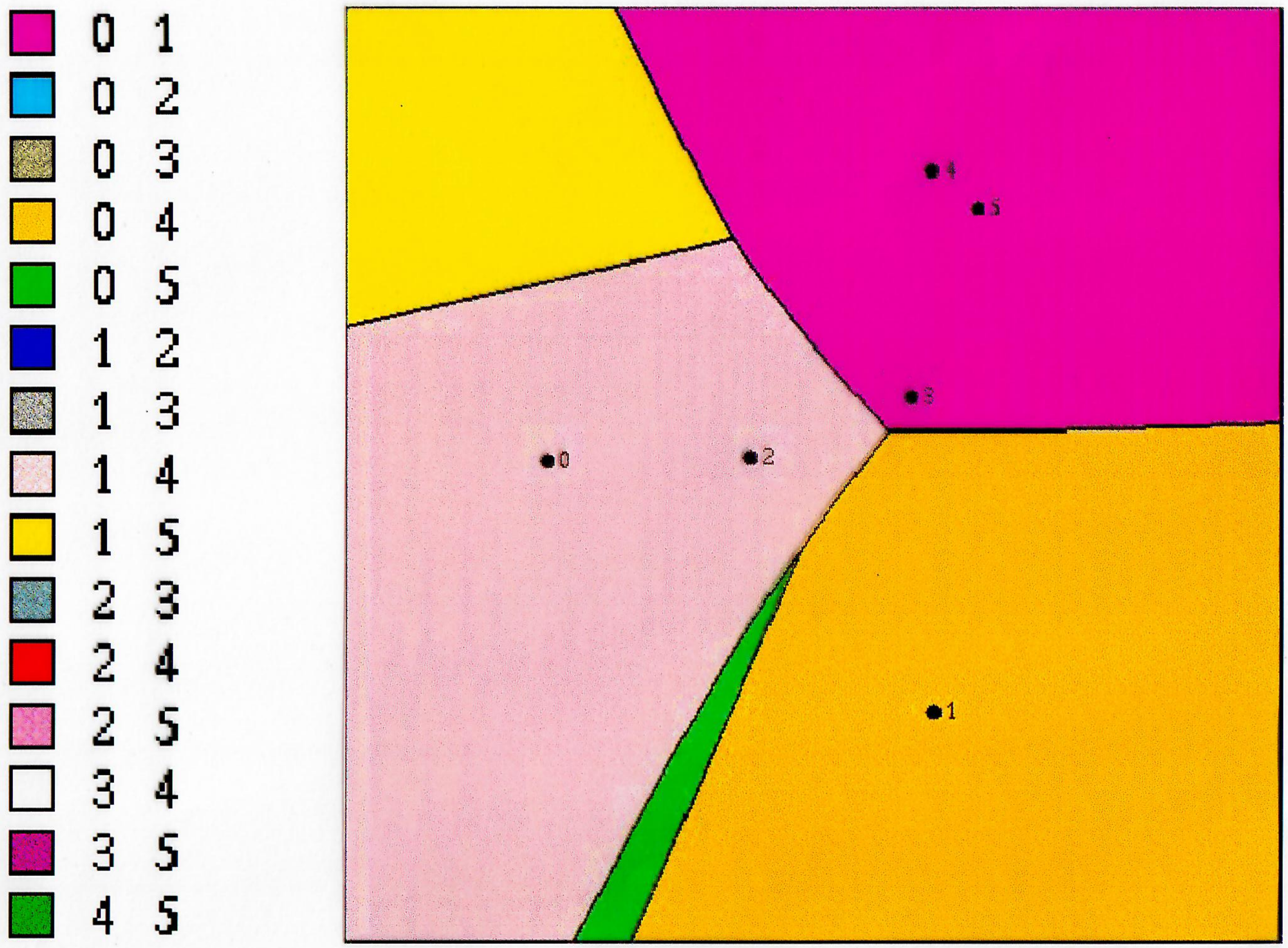
Upper bound: lower envelope of algebraic surfaces of constant degree [Sharir-Agarwal 95].

Theorem: $V_{\mathcal{D}}^{(F)}(S) = \Theta(n^2)$.

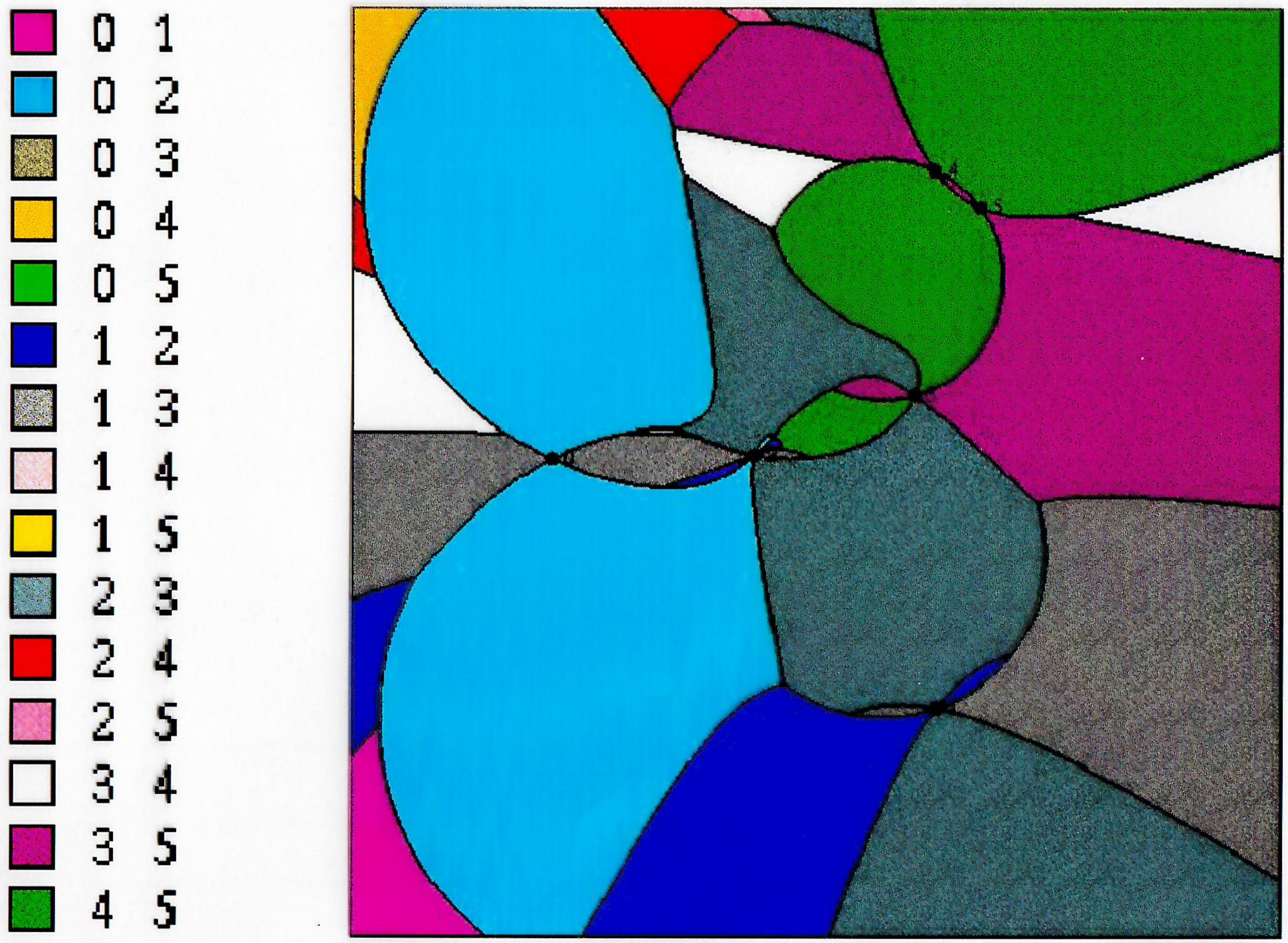
Proof: Overlay of the regular $V^{(N|F)}(S)$.



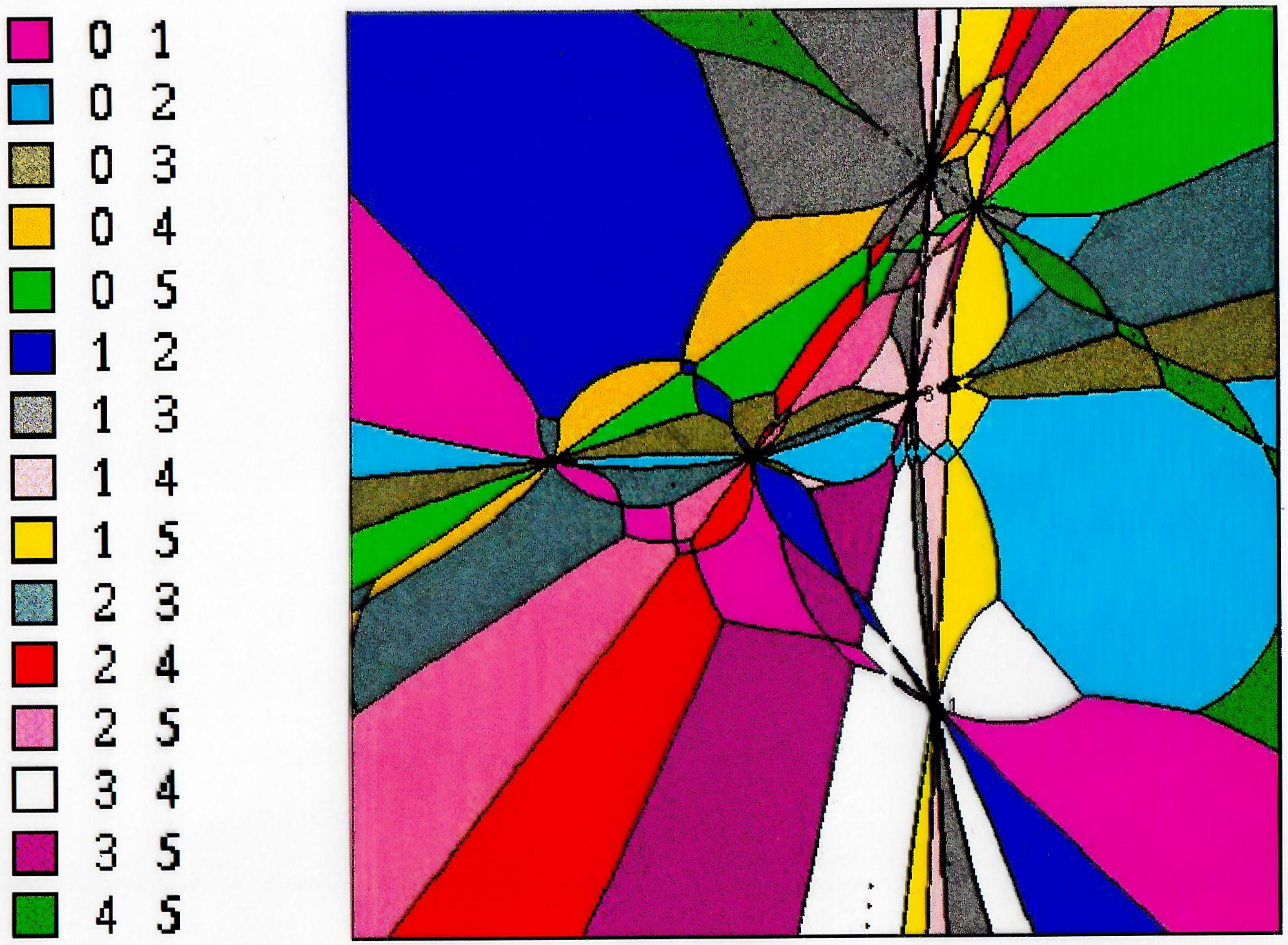
Triangle perimeter: NN Voronoi diag.



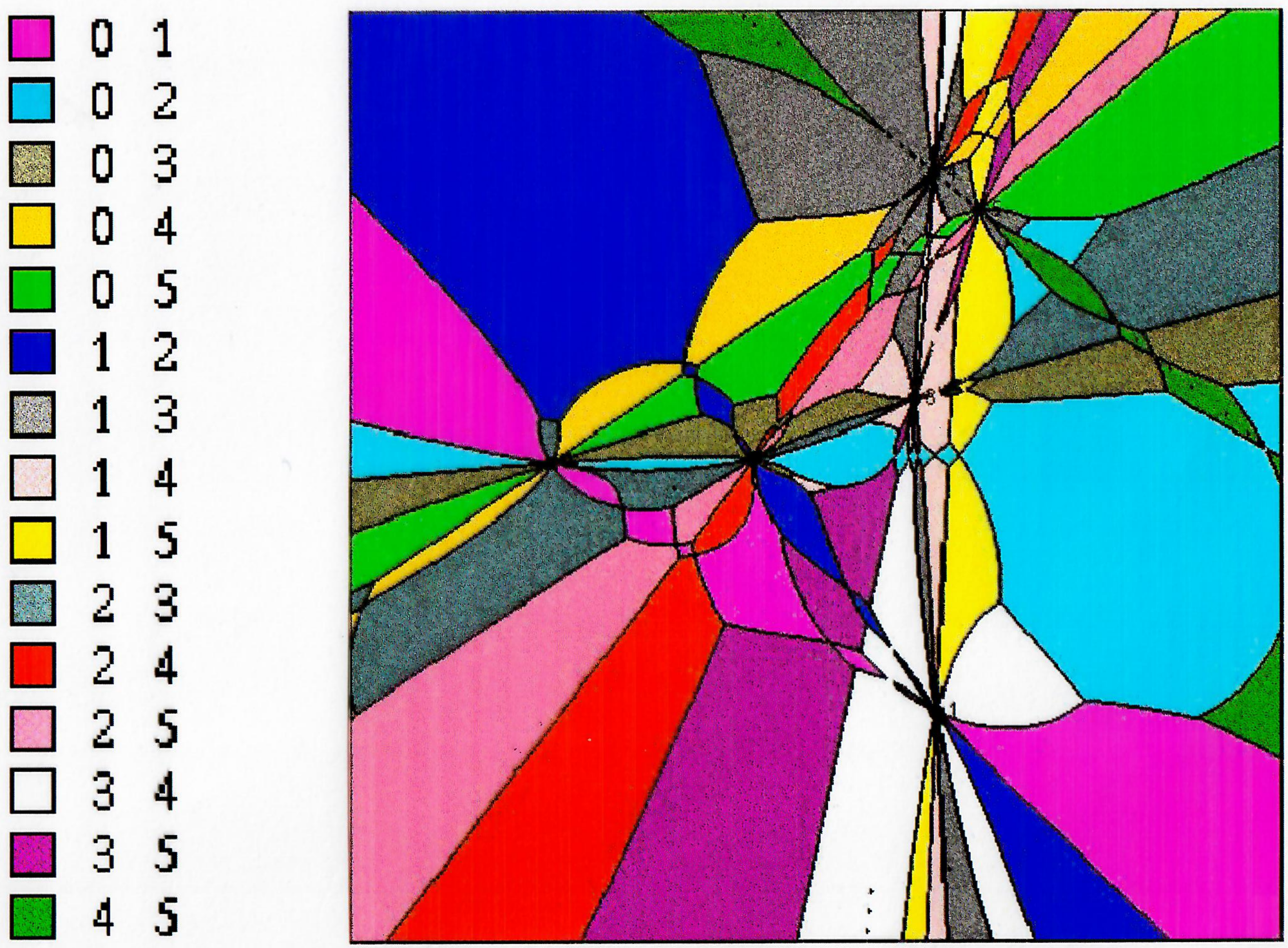
Triangle perimeter: FN Voronoi diag.



Circumcircle radius: NN Voronoi diag.



Circumcircle radius: FN Voronoi diag.



Circumcircle radius: FN Voronoi diag.

Java Applet

Implementing bisector/NN/FN for all the discussed distance functions.

Look at:

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http://www.middlebury.edu/
   ~dickerso/research/dfunct.html
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or

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http://www.cgc.cs.jhu.edu/
   ~barequet/2-point.html
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