On the Crossing-Number Lemma

(minutes from a CG class, Gill Barequet)

Definition 1 *The* crossing number *of a graph G, denoted by $\#cr(G)$ ($\#c$ in short), is the minimum possible number of edge crossings in a planar drawing of G.*

Lemma 1 ([ACNS, Le]) *Let G be a* simple *graph (no parallel edges, no self edges) with n vertices and $e \geq 4n$ edges. Then $\#cr(G) = \Omega(e^3/n^2)$.*

Application 1. Assume we are given a set P of points, and a set L of lines in the plane. The number of *incidences* between P and L, denoted as $\#i(P,L)$ ($\#i$ in short), is the number of pairs $p \in P, l \in L$, such that p is incident to l.

Theorem 2 *Given a set P of n points, and a a set L of ℓ lines in the plane, $\#i(P,L) = O((n\ell)^{2/3} + n + \ell)$.*

Proof: Assume without loss of generality that there is no line in L that is not incident to any point in P. Define a graph G whose vertex set is P. Connect by an edge two vertices of G if they are consecutive along any line of L. Obviously $\#cr(G) \leq \ell^2$. Each line $l \in L$ contributes $k_l - 1$ edges to G, where k_l is the number of points of L incident to l. Summing up for all lines in L we get $e = \#i - \ell$, where e is the number of edges in G. Distinguish between two cases:

1. $e < 4n$. In this case $\#i - \ell < 4n$, that is, $\#i < 4n + \ell$.

2. $e \geq 4n$. In this case we apply the crossing-number lemma to obtain $\#c = \Omega(e^3/n^2) = \Omega((\#i - \ell)^3/n^2)$. On the other hand, $\#c = O(\ell^2)$. The combination of the lower and upper bounds on $\#c$ yields $(\#i - \ell)^3 = O(n^2\ell^2)$, that is, $\#i = O((n\ell)^{2/3} + \ell)$.

The combination of the two cases completes the proof. □

Application 2. Let $A \subset \mathbb{Z}$ be a set of n integral numbers. Construct the set $B = A \cdot A + A$. Obviously $k = |B| = O(n^3)$.

Theorem 3 $k = \Omega(n^{3/2})$.

Proof: Define a set of points $S = \{(x,y) | x \in A, y \in B\}$. Obviously $|S| = nk$. Draw all the lines $y = a_i x + a_j$, where $a_i, a_j \in A$. There are n^2 such lines. Since each of these lines is incident to n points of S (verify!), we have $\#i = n^3$. According the first application,

$$n^3 = \#i = O((nk \cdot n^2)^{2/3} + nk + n^2) = O(n^2 k^{2/3} + nk + n^2).$$

The last term is redundant since it is smaller than the first term. In addition, $k \leq n^3$ implies $kn \leq k^{2/3} n^2$, so the second term is also redundant. Therefore $n^3 = O(n^2 k^{2/3})$, that is, $k = \Omega(n^{3/2})$, and the claim follows. □

References

[ACNS] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, Crossing-free subgraphs, *Annals of Discrete Mathematics*, 12 (1982), 9–12.

[Le] F.T. Leighton, Complexity Issues in VLSI, *MIT Press*, Cambridge, MA, 1983.